ELEVENTH EDITION

Finite Mathematics

Lial • Greenwell • Ritchey

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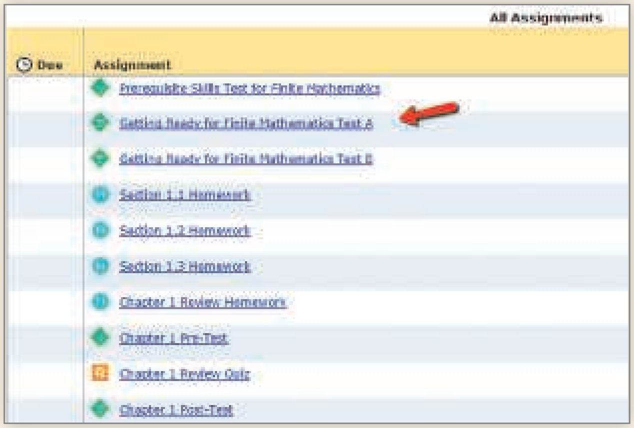

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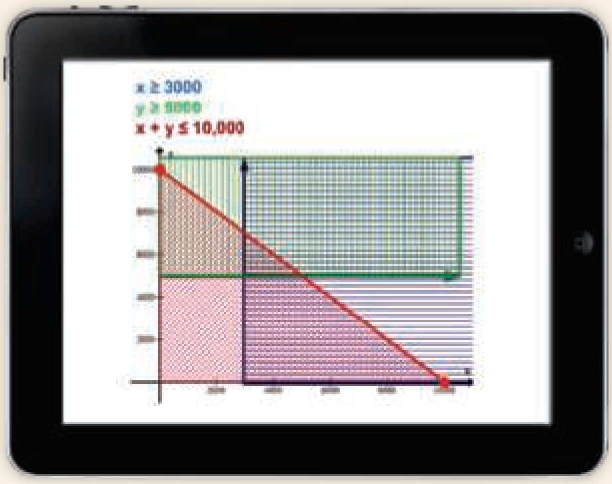

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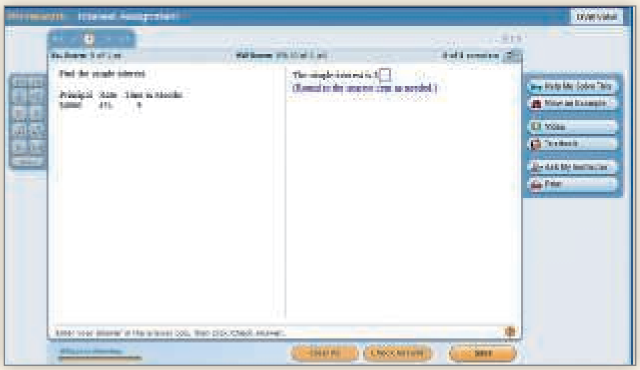

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Margaret L. Lial
American River College

Raymond N. Greenwell
Hofstra University

Nathan P. Ritchey
Edinboro University

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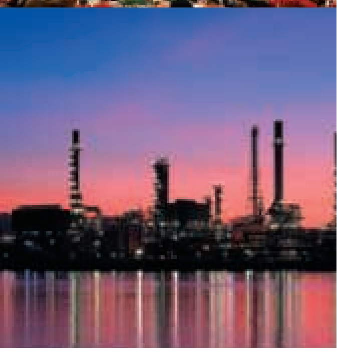


Appendix

Special Topics to Accompany Finite Mathematics

The following material is provided free to adopters at www.pearsonhighered.com/mathstatsresources:

Digraphs and Networks

Preface

Finite Mathematics is a thorough, applications-oriented text for students majoring in business, management, economics, or the life or social sciences. In addition to its clear exposition, this text consistently connects the mathematics to career and everyday-life situations. A prerequisite of two or three semesters of high school algebra is assumed. A greatly enhanced MyMathLab course, new applications and exercises, and other new learning tools make this 11th edition an even richer learning resource for students.

Our Approach

Our main goal is to present finite mathematics in a concise and meaningful way so that students can understand the full picture of the concepts they are learning and apply them to real-life situations. This is done through a variety of means.

Focus on Applications Making this course meaningful to students is critical to their success. Applications of the mathematics are integrated throughout the text in the exposition, the examples, the exercise sets, and the supplementary resources. We are constantly on the lookout for novel applications, and the text reflects our efforts to infuse it with relevance. Our research is showcased in the Index of Applications at the back of the book and the extended list of sources of real-world data on www.pearsonhighered.com/mathstatsresources. *Finite Mathematics* presents students with myriad opportunities to relate what they're learning to career situations through the *Apply It* question at the beginning of sections, the applied examples and exercises, and the *Extended Application* at the end of each chapter.

Pedagogy to Support Students Students need careful explanations of the mathematics along with examples presented in a clear and consistent manner. Additionally, students and instructors should have a means to assess the basic prerequisite skills needed for the course content. This can be done with the *Prerequisite Skills Diagnostic Test*, located just prior to Chapter R. If the diagnostic test reveals gaps in basic skills, students can find help right within the text. Further, *Warm-Up Exercises* are now included at the beginning of many exercise sets. Within MyMathLab are additional diagnostic tests (one per chapter), and remediation is automatically personalized to meet student needs. Students will appreciate the many annotated examples within the text, the *Your Turn* exercises that follow examples, the *For Review* references, and the wealth of learning resources within MyMathLab.

Beyond the Textbook Students today want resources at their fingertips and, for them, that means digital. So Pearson has developed a robust MyMathLab course for *Finite Mathematics*. MyMathLab has a well-established and well-documented track record of helping students succeed in mathematics. The MyMathLab online course for this text contains over 3600 exercises to challenge students and provides help when they need it. Students who learn best through video can view (and review) section- and example-level videos within MyMathLab. These and other resources are available to students as a unified and reliable tool for their success.

New to the Eleventh Edition

Based on our experience in the classroom along with feedback from many instructors across the country, the focus of this revision is to improve the clarity of the presentation and provide students with more opportunities to learn, practice, and apply what they've learned on their own. This is achieved both in the presentation of the content and in the new features added to the text.

New Features

- *Warm-Up Exercises* were added to many exercise sets to provide an opportunity for students to refresh key prerequisite skills at "point of use."
- Graphing calculator screens have been updated to reflect the TI-84 Plus C, which features color and a much higher screen resolution. Additionally, the graphing calculator notes have been updated throughout.
- We added more "help text" annotations to examples. These notes, set in small blue type, appear next to the steps within worked-out examples and provide an additional aid for students with weaker algebra skills.
- For many years this text has featured enormous amounts of real data used in examples and exercises. The 11th edition will not disappoint in this area. We have added or updated 107 (12.5%) of the application exercises throughout the text.
- We updated exercises and examples based on user feedback and other factors. Of the 2646 exercises within the sections, 208 (7.8%) are new or updated. Of the 318 examples, 64 (20.1%) are new or updated.
- MyMathLab contains a wealth of new resources to help students learn and help you as you teach. Some resources were added or revised based on student usage of the *previous* edition of the MyMathLab course. For example, more exercises were added to those chapters and sections that are more widely assigned.
    - Hundreds of new exercises were added to the course to provide you with more options for assignments, including:
        - More application exercises throughout the text
        - *Setup & Solve* exercises that require students to specify how to set up a problem as well as solve it
        - Exercises that take advantage of the enhanced graphing tool
    - An Integrated Review version of the course contains preassigned diagnostic and remediation resources for key prerequisite skills. Skills Check Quizzes help diagnose gaps in skills prior to each chapter. MyMathLab then provides personalized help on only those skills that a student has not mastered.
    - The videos for the course have increased in number, type, and quality:
        - New videos feature more applications and more challenging examples.
        - In addition to full-length lecture videos, MyMathLab now includes assignable, shorter video clips that focus on a specific concept or example.
        - Tutorial videos involving graphing calculators are available to augment existing "by hand" methods, allowing you to tailor the help that students receive to how you incorporate graphing calculators into the course.
        - A Guide to Video-Based Instruction shows which exercises correspond to each video, making it easy to assess students after they watch an instructional video. This is perfect for flipped-classroom situations.
    - Learning Catalytics is a "bring your own device" student engagement, assessment, and classroom intelligence system. Students use any modern web-enabled device they already have—laptop, smartphone, or tablet. With Learning Catalytics, you assess students in real time, using open-ended tasks to probe student understanding. It allows you to engage students by creating open-ended questions that ask for numerical, algebraic, textual, or graphical responses—or just plain multiple-choice. Students who have access to MyMathLab have instant access to Learning Catalytics and can log in using their MyMathLab username and password. Learning Catalytics contains Pearson-created content for finite mathematics that allows you to take advantage of this exciting technology immediately.

New and Revised Content

The chapters and sections in the text are in the same order as the previous edition, making it easier for users to transition to the new edition. In addition to revising exercises and examples throughout, updating and adding real-world data, we made the following changes:

Chapter R

- Added new *Your Turn* exercises to ensure that there is a student assessment for each major concept.
- Added more detail to R.2 on factoring perfect squares.

Chapter 1

- Rewrote the part of 1.1 involving graphing lines, emphasizing different methods for graphing.
- Rewrote 1.2 on supply, demand, break-even analysis, and equilibrium; giving formal definitions that match what students would see in business and economics courses. All of the business applications were revised, according to recommendations from reviewers, to be more in line with business texts. Also added a new Example 6 on finding a cost function.
- Added color for pedagogical reasons to make content easier to follow.

Chapter 2

- In 2.1, added a new definition for consistent systems. Also added definitions of general solution, parameter, and particular solution and explanations of how to find each.
- In 2.5, added a shortcut for finding the inverse of a 2×2 matrix.

Chapter 3

- Revised two Technology Notes in 3.1, giving more details on how to graph an inequality and how to add color. Also added a new Example 6, which illustrates what happens when there is no feasible region.
- In 3.2, revised a Technology Note explaining how to find the points of intersection in the feasible region.

Chapter 4

- Added column headings to 4.1 to explain where basic variables are. Additionally, revised Example 4 to include more explanation and detail.
- Added a Caution note to 4.2 and revised the Technology Note for using Excel to solve linear programming problems.

Chapter 5

- In 5.1, revised Example 2 to add more detail regarding what each variable designates. Revised Examples 4–6 and 8–10 to better explain concepts and/or update real-world data. Revised Example 12 to include more explanation—important because we gave two methods to find compounding time (a graphing calculator method and an optional method that uses logarithms).
- Added new Technology Notes to 5.1, 5.2, and 5.3, explaining how to use the TVM Solver in the TI-84.
- Rewrote the 5.2 introduction and converted it into a new example on annuities. Also added a note on why particular variables were chosen and that these variables may look different in other places. Revised Examples 4 and 5 for content and added technology.
- In 5.3, revised Example 5 to better explain amortization schedules.

Chapter 6

- Updated and added numerous exercises.

Chapter 7

- Updated Examples in 7.1, 7.2, and 7.4 for content, data, and/or clarity.
- Added more explanation of independence to 7.5.
- Completely rewrote the Extended Application at the end of the chapter.

Chapter 8

- Updated and added numerous exercises as well as an example in 8.5.

Chapter 9

- Added Technology Notes explaining how to use the calculator to find probabilities, how to create a histogram, and so on.
- Reorganized 9.1 into two parts (frequency distributions and central tendency) and added new headings for mean, median, and mode. Also added a new Example 9, comparing mean and median when outliers exist.
- Switched the emphasis of 9.3 from using the table in the back of the book to using a calculator. (We still use the table but all of the answers are obtained using a calculator.) In that vein, we explained in great detail how to use the calculator to find the various probabilities, and changed Examples 1, 2, and 4.
- Rewrote the introduction to 9.4 as an example and converted other exposition into a second example to better illustrate the concepts being described. Added a Technology Note to the second new example.

Chapter 10

- Added more explanation of powers of the transition matrix in 10.1.

Chapter 11

- Revised the introductory material of 11.1 to create a new Example 1.
- Added more explanation to Example 3 in 11.2.

Features of *Finite Mathematics*

Chapter Opener

Each chapter opens with a quick introduction that relates to an application presented in the chapter.

Apply It

An Apply It question, typically at the start of a section, motivates the math content of the section by posing a real-world question that is then answered within the examples or exercises.

Teaching Tips

Teaching Tips are provided in the margins of the Annotated Instructor's Edition for those who are new to teaching this course.

For Review

For Review boxes are provided in the margin as appropriate, giving students just-in-time help with skills they should already know but may have forgotten. For Review comments sometimes include an explanation, while others refer students back to earlier parts of the book for a more thorough review.

FOR REVIEW

Recall that I is the identity matrix, a square matrix in which each element on the main diagonal is 1 and all other elements are 0.

Caution

Caution notes provide students with a quick "heads up" to common difficulties and errors.

CAUTION If there is a 0 in the right-hand column, do not disregard that row, unless the corresponding number in the pivot column is negative or zero. In fact, such a row gives a quotient of 0, so it will automatically have the smallest ratio. It will not cause an increase in z, but it may lead to another tableau in which z can be further increased.

Your Turn Exercises

These exercises follow selected examples and provide students with an easy way to quickly stop and check their understanding. Answers are provided at the end of the section's exercises.

Technology Notes

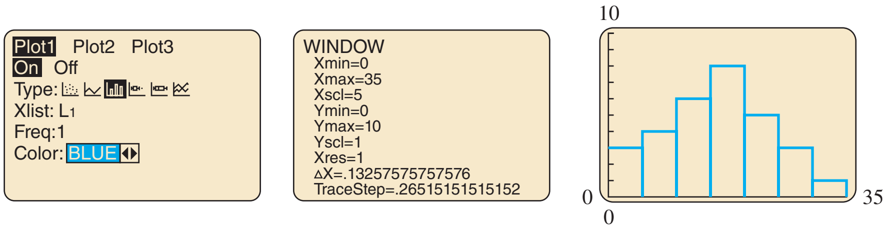
Material on graphing calculators or Microsoft Excel is clearly labeled to make it easier for instructors to use this material (or not).

- **New** The figures depicting calculator screens now reflect the TI-84 Plus C, which features color and higher pixel counts.

TECHNOLOGY NOTE

Many graphing calculators have the capability of drawing a histogram. On a TI-84 Plus C calculator, enter the data by selecting `STAT EDIT` and then typing the entries in L_1. To create a histogram, select `STATPLOT` and `1:Plot1`. Turn the plot `ON` and select the histogram icon in the `Type` row, as shown in Figure 2(a). Be sure that L_1 is entered for `Xlist`.

Select `WINDOW` to set up the intervals, as shown in Figure 2(b). The `Xscl` value gives the width of each interval. To draw the histogram, select `GRAPH`. Figure 2(c) shows the histogram for the data of Example 1. See the *Graphing Calculator and Excel Spreadsheet Manual* available with this text.



(a) (b) (c)

FIGURE 2

Exercise Sets

Basic exercises are followed by an Applications section, which is grouped by subheads such as "Business and Economics." Other types of exercises include the following:

- **New** **Warm-Up Exercises** at the beginning of most sections provide a chance for students to refresh the key prerequisite skills needed for the section's exercises.
- **Connections** exercises integrate topics presented in different sections or chapters and are indicated with .
- **Technology** exercises are labeled for graphing calculator and for spreadsheets.
- **Writing** exercises, labeled with , provide students with an opportunity to explain important mathematical ideas.
- Exercises that are particularly challenging are denoted with a + in the Annotated Instructor's Edition only.
- The Annotated Instructor's Edition contains most answers right on the page. Overflow answers are at the back of the book.

Chapter Summary and Review

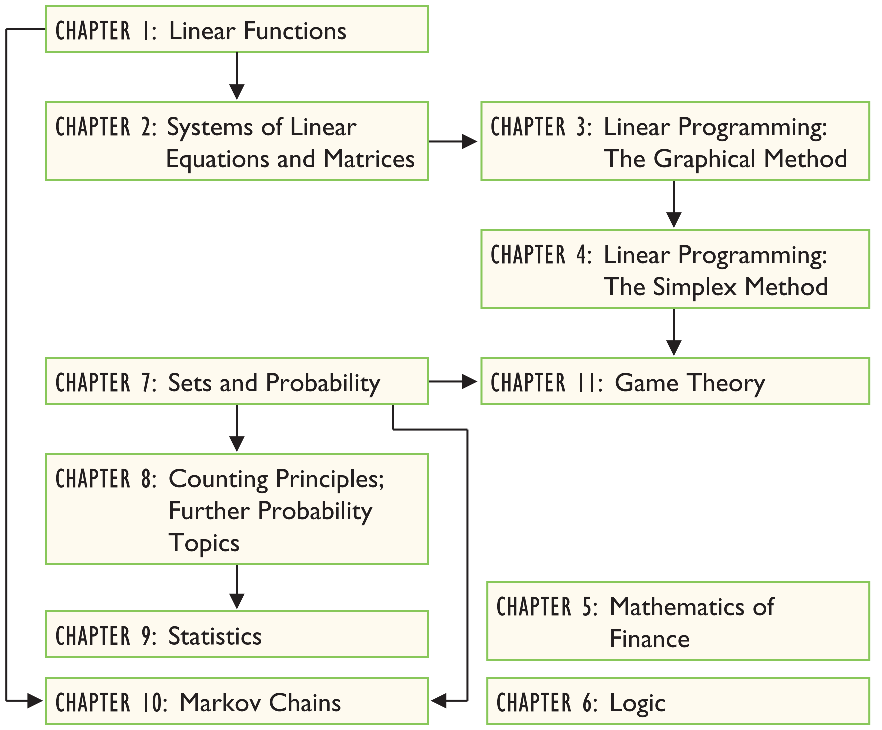
- The end-of-chapter **Summary** provides students with a quick summary of the key ideas of the chapter followed by a list of key definitions, terms, and examples.
- Chapter **Review Exercises** include Concept Check exercises and an ample set of Practice and Exploration exercises. This arrangement provides students with a comprehensive set of exercises to prepare for chapter exams.

Extended Applications

- Extended Applications are provided at the end of every chapter as in-depth applied exercises to help stimulate student interest. These activities can be completed individually or as group projects.

Flexible Syllabus

The flexibility of the text is indicated in the following chart of chapter prerequisites. As shown, the course could begin with either Chapter 1 or Chapter 7. Chapters 5 and 6 could be covered at any time, although Chapter 6 makes a nice introduction to Chapter 7.



Supplements

FOR STUDENTS	FOR INSTRUCTORS

Student's Solutions Manual (in print and electronically within MyMathLab)

- Provides detailed solutions to all odd-numbered text exercises and sample chapter tests with answers.
- ISBN 0321997425 / 9780321997425

Supplementary Content (online only)

- Chapter on Digraphs and Networks
- Additional Extended Applications
- Comprehensive Source List
- Available at the Downloadable Student Resources site, www.pearsonhighered.com/mathstatsresources, and within MyMathLab

Annotated Instructor's Edition

- Includes all the answers, usually on the same page as the exercises, for quick reference
- Challenge problems labeled with +
- Numerous teaching tips
- ISBN 0321998782 / 9780321998781

Instructor's Solutions and Resource Manual (Download Only)

- Provides complete solutions to all exercises, two versions of a pre-test and final exam, and teaching tips.
- Available to qualified instructors within MyMathLab or through the Pearson Instructor Resource Center (www.pearsonhighered.com/irc).
- ISBN 0133864499 / 9780133864496

FOR STUDENTS	FOR INSTRUCTORS

Graphing Calculator Manual for Applied Mathematics (downloadable)

- Contains detailed instruction for using the Tl-83/ TI-83+/ T1-84+C
- Instructions are organized by topic.
- Downloadable from within MyMathLab

Excel Spreadheet Manual for Applied Mathematics (downloadable)

- Contains delailed instruction for using Excel 2013
- Instructions are organized by topic.
- Downloadable from within MyMathLab

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- Includes lecture content and key graphics from the book.
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TestGen Computerized Test Bank

- TestGen® (www.pearsoned.com/testgen) enables instructors to build, edit, print, and administer tests using a computerized bank of questions developed to cover all the objectives of the text.
- TestGen is algorithmically based, allowing instructors to create multiple but equivalent versions of the same question or test with the click of a button. Instructors can also modify test bank questions or add new questions.
- The software and testbank are available to qualified instructors within MyMathLab or through the Pearson Instructor Resource Center (www.pearsonhighered .com/irc).
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MyMathLab® Online Course (access code required)

MyMathLab delivers **proven results** in helping individual students succeed.

- MyMathLab has a consistently positive impact on the quality of learning in higher education math instruction. MyMathLab can be successfully implemented in any environment—lab-based, hybrid, fully online, traditional—and demonstrates the quantifiable difference that integrated usage has on student retention, subsequent success, and overall achievement.
- MyMathLab's comprehensive online gradebook automatically tracks your students' results on tests, quizzes, homework, and in the study plan. You can use the gradebook to quickly intervene if your students have trouble, or to provide positive feedback on a job well done. The data within MyMathLab's gradebook are easily exported to a variety of spreadsheet programs.

MyMathLab provides **engaging experiences** that personalize, stimulate, and measure learning for each student.

- **Personalized Learning:** MyMathLab offers two main features that support adaptive learning: personalized homework and the adaptive study plan. These features allow your students to work on just what they need to learn when it makes the most sense, maximizing their potential for understanding and success.
- **Exercises:** The homework and practice exercises in MyMathLab are correlated to the exercises in the textbook, and they regenerate algorithmically to give students unlimited opportunity for practice and mastery. The software provides helpful feedback when students enter incorrect answers and includes optional learning aids, including guided solutions, sample problems, animations, videos, and eText.

- **Learning and Teaching Tools** include:
  - **Learning Catalytics**—a "bring your own device" student engagement, assessment, and classroom intelligence system, included within MyMathLab. Includes questions written specifically for this course.
  - **Instructional videos—**full-length lecture videos as well as shorter example-based videos.
  - **Help for Gaps in Prerequisite Skills—**diagnostic quizzes tied to personalized assignments help address gaps in algebra skills that might otherwise impede success.
  - **Excel Spreadsheet Manual**—specifically written for this course.
  - **Graphing Calculator Manual**—specifically written for this course.
- **Complete eText** is available to students through MyMathLab courses for the lifetime of the edition, giving students unlimited access to the eBook within any course using that edition of the textbook.
- **MyMathLab Accessibility:** MyMathLab is compatible with the JAWS screen reader, which enables multiple-choice and free-response problem types to be read and interacted with via keyboard controls and math notation input. MyMathLab also works with screen enlargers, including ZoomText, MAGic, and SuperNova. And all MyMathLab videos have closed captioning. More information on this functionality is available at http://mymathlab.com/accessibility.

MyMathLab comes from an **experienced partner** with educational expertise and an eye on the future.

- Knowing that you are using a Pearson product means knowing that you are using quality content. This means that our eTexts are accurate and our assessment tools work. It means we are committed to making MyMathLab as accessible as possible.
- Whether you are just getting started with MyMathLab, or have a question along the way, we're here to help you learn about our technologies and how to incorporate them into your course.
- To learn more about how MyMathLab combines proven learning applications with powerful assessment and continuously adaptive capabilities, visit **www.mymathlab.com** or contact your Pearson representative.

Additional Online Courseware Options (access code required)

- **MathXL®** is the homework and assessment engine that runs MyMathLab. (MyMathLab is MathXL plus a learning management system.)
- **MyMathLab Integrated Review Course** contains preassigned diagnostic and remediation resources for key prerequisite skills. Skills Check Quizzes help diagnose gaps in skills prior to each chapter. MyMathLab then provides personalized help on just those skills that a student has not mastered.
- **MyMathLab® Plus** contains all the features of MyMathLab with convenient management tools and a dedicated services team. It includes batch enrollment so everyone can be ready to start class on the first day, login directly from your campus portal, advanced reporting features, and 24/7 support by email or online chat.

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Raymond N. Greenwell
Nathan P. Ritchey

Prerequisite Skills Diagnostic Test

Below is a very brief test to help you recognize which, if any, prerequisite skills you may need to remediate in order to be successful in this course. After completing the test, check your answers in the back of the book. In addition to the answers, we have also provided the solutions to these problems in Appendix A. These solutions should help remind you how to solve the problems. For problems 5–10, the answers are followed by references to sections within Chapter R where you can find guidance on how to solve the problem and/or additional instruction. Addressing any weak prerequisite skills now will make a positive impact on your success as you progress through this course.

1. What percent of 50 is 10?

2. Simplify $\frac{13}{7} - \frac{2}{5}$.

3. Let x be the number of apples and y be the number of oranges. Write the following statement as an algebraic equation: "The total number of apples and oranges is 75."

4. Let s be the number of students and p be the number of professors. Write the following statement as an algebraic equation: "There are at least four times as many students as professors."

5. Solve for k: $7k + 8 = -4(3 - k)$.

6. Solve for x: $\frac{5}{8}x + \frac{1}{16}x = \frac{11}{16} + x$.

7. Write in interval notation: $-2 < x \leq 5$.

8. Using the variable x, write the following interval as an inequality: $[-\infty, -3]$.

9. Solve for y: $5(y - 2) + 1 \leq 7y + 8$.

10. Solve for p: $\frac{2}{3}(5p - 3) > \frac{3}{4}(2p + 1)$.

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R Algebra Reference

In this chapter, we will review the most important topics in algebra. Knowing algebra is a fundamental prerequisite to success in higher mathematics. This algebra reference is designed for self-study; study it all at once or refer to it when needed throughout the course. Since this is a review, answers to all exercises are given in the answer section at the back of the book.

R.1 Polynomials

An expression such as $9p^4$ is a **term**; the number 9 is the **coefficient**, p is the **variable**, and 4 is the **exponent**. The expression p^4 means $p \cdot p \cdot p \cdot p$, while p^2 means $p \cdot p$, and so on. Terms having the same variable and the same exponent, such as $9x^4$ and $-3x^4$, are **like terms**. Terms that do not have both the same variable and the same exponent, such as m^2 and m^4, are **unlike terms**.

A **polynomial** is a term or a finite sum of terms in which all variables have whole number exponents, and no variables appear in denominators. Examples of polynomials include

$$5x^4 + 2x^3 + 6x, \qquad 8m^3 + 9m^2n - 6mn^2 + 3n^3, \qquad 10p, \qquad \text{and} \qquad -9.$$

Order of Operations

Algebra is a language, and you must be familiar with its rules to correctly interpret algebraic statements. The following order of operations has been agreed upon through centuries of usage.

- Expressions in **parentheses** (or other grouping symbols) are calculated first, working from the inside out. The numerator and denominator of a fraction are treated as expressions in parentheses.
- **Powers** are performed next, going from left to right.
- **Multiplication** and **division** are performed next, going from left to right.
- **Addition** and **subtraction** are performed last, going from left to right.

For example, in the expression $[6(x + 1)^2 + 3x - 22]^2$, suppose x has the value of 2. We would evaluate this as follows:

$$\begin{aligned}
[6(2 + 1)^2 + 3(2) - 22]^2 &= [6(3)^2 + 3(2) - 22]^2 && \text{Evaluate the expression in the innermost parentheses.}\\
&= [6(9) + 3(2) - 22]^2 && \text{Evaluate 3 raised to a power.}\\
&= (54 + 6 - 22)^2 && \text{Perform the multiplications.}\\
&= (38)^2 && \text{Perform the addition and subtraction from left to right.}\\
&= 1444 && \text{Evaluate the power.}
\end{aligned}$$

In the expression $\dfrac{x^2 + 3x + 6}{x + 6}$, suppose x has the value of 2. We would evaluate this as follows:

$$\begin{aligned}
\frac{2^2 + 3(2) + 6}{2 + 6} &= \frac{16}{8} && \text{Evaluate the numerator and the denominator.}\\
&= 2 && \text{Simplify the fraction.}
\end{aligned}$$

Adding and Subtracting Polynomials

The following properties of real numbers are useful for performing operations on polynomials.

Properties of Real Numbers

For all real numbers a, b, and c:

1. $a + b = b + a$; $ab = ba$;	**Commutative properties**
2. $(a + b) + c = a + (b + c)$; $(ab)c = a(bc)$;	**Associative properties**
3. $a(b + c) = ab + ac$.	**Distributive property**

EXAMPLE 1 Properties of Real Numbers

(a) $2 + x = x + 2$ — **Commutative property of addition**

(b) $x \cdot 3 = 3x$ — **Commutative property of multiplication**

(c) $(7x)x = 7(x \cdot x) = 7x^2$ — **Associative property of multiplication**

(d) $3(x + 4) = 3x + 12$ — **Distributive property**

One use of the distributive property is to add or subtract polynomials. Only like terms may be added or subtracted. For example,

$$12y^4 + 6y^4 = (12 + 6)y^4 = 18y^4,$$

and

$$-2m^2 + 8m^2 = (-2 + 8)m^2 = 6m^2,$$

but the polynomial $8y^4 + 2y^5$ cannot be further simplified. To subtract polynomials, we use the facts that $-(a + b) = -a - b$ and $-(a - b) = -a + b$. In the next example, we show how to add and subtract polynomials.

EXAMPLE 2 Adding and Subtracting Polynomials

Add or subtract as indicated.

(a) $(8x^3 - 4x^2 + 6x) + (3x^3 + 5x^2 - 9x + 8)$

SOLUTION Combine like terms.

$$\begin{aligned}(8x^3 - 4x^2 + 6x) &+ (3x^3 + 5x^2 - 9x + 8)\\ &= (8x^3 + 3x^3) + (-4x^2 + 5x^2) + (6x - 9x) + 8\\ &= 11x^3 + x^2 - 3x + 8\end{aligned}$$

(b) $2(-4x^4 + 6x^3 - 9x^2 - 12) + 3(-3x^3 + 8x^2 - 11x + 7)$

SOLUTION Multiply each polynomial by the factor in front of the polynomial, and then combine terms as before.

$$\begin{aligned}2(-4x^4 + 6x^3 - 9x^2 - 12) &+ 3(-3x^3 + 8x^2 - 11x + 7)\\ &= -8x^4 + 12x^3 - 18x^2 - 24 - 9x^3 + 24x^2 - 33x + 21\\ &= -8x^4 + 3x^3 + 6x^2 - 33x - 3\end{aligned}$$

(c) $(2x^2 - 11x + 8) - (7x^2 - 6x + 2)$

SOLUTION Distributing the minus sign and combining like terms yields

$$\begin{aligned}(2x^2 - 11x + 8) &+ (-7x^2 + 6x - 2)\\ &= -5x^2 - 5x + 6.\end{aligned}$$

TRY YOUR TURN 1

YOUR TURN 1 Perform the operation $3(x^2 - 4x - 5) - 4(3x^2 - 5x - 7)$.

Multiplying Polynomials

The distributive property is also used to multiply polynomials, along with the fact that $a^m \cdot a^n = a^{m+n}$. For example,

$$x \cdot x = x^1 \cdot x^1 = x^{1+1} = x^2 \qquad \text{and} \qquad x^2 \cdot x^5 = x^{2+5} = x^7.$$

EXAMPLE 3 Multiplying Polynomials

Multiply.

(a) $8x(6x - 4)$

SOLUTION Using the distributive property yields

$$\begin{aligned}8x(6x - 4) &= 8x(6x) - 8x(4)\\ &= 48x^2 - 32x.\end{aligned}$$

(b) $(3p - 2)(p^2 + 5p - 1)$

SOLUTION Using the distributive property yields

$$\begin{aligned}
&(3p - 2)(p^2 + 5p - 1)\\
&\quad = 3p(p^2 + 5p - 1) - 2(p^2 + 5p - 1)\\
&\quad = 3p(p^2) + 3p(5p) + 3p(-1) - 2(p^2) - 2(5p) - 2(-1)\\
&\quad = 3p^3 + 15p^2 - 3p - 2p^2 - 10p + 2\\
&\quad = 3p^3 + 13p^2 - 13p + 2.
\end{aligned}$$

(c) $(x + 2)(x + 3)(x - 4)$

SOLUTION Multiplying the first two polynomials and then multiplying their product by the third polynomial yields

$$\begin{aligned}
&(x + 2)(x + 3)(x - 4)\\
&\quad = [(x + 2)(x + 3)](x - 4)\\
&\quad = (x^2 + 2x + 3x + 6)(x - 4)\\
&\quad = (x^2 + 5x + 6)(x - 4)\\
&\quad = x^3 - 4x^2 + 5x^2 - 20x + 6x - 24\\
&\quad = x^3 + x^2 - 14x - 24.
\end{aligned}$$

TRY YOUR TURN 2

YOUR TURN 2 Perform the operation $(3y + 2)(4y^2 - 2y - 5)$.

A **binomial** is a polynomial with exactly two terms, such as $2x + 1$ or $m + n$. When two binomials are multiplied, the FOIL method (First, Outer, Inner, Last) is used as a memory aid.

EXAMPLE 4 Multiplying Polynomials

Find $(2m - 5)(m + 4)$ using the FOIL method.

SOLUTION

$$\begin{aligned}
(2m - 5)(m + 4) &= \overset{\text{F}}{(2m)(m)} + \overset{\text{O}}{(2m)(4)} + \overset{\text{I}}{(-5)(m)} + \overset{\text{L}}{(-5)(4)}\\
&= 2m^2 + 8m - 5m - 20\\
&= 2m^2 + 3m - 20
\end{aligned}$$

TRY YOUR TURN 3

YOUR TURN 3 Find $(2x + 7)(3x - 1)$ using the FOIL method.

EXAMPLE 5 Multiplying Polynomials

Find $(2k - 5m)^3$.

SOLUTION Write $(2k - 5m)^3$ as $(2k - 5m)(2k - 5m)(2k - 5m)$. Then multiply the first two factors using FOIL.

$$\begin{aligned}
(2k - 5m)(2k - 5m) &= 4k^2 - 10km - 10km + 25m^2\\
&= 4k^2 - 20km + 25m^2
\end{aligned}$$

Now multiply this last result by $(2k - 5m)$ using the distributive property, as in Example 3(c).

$$\begin{aligned}
&(4k^2 - 20km + 25m^2)(2k - 5m)\\
&\quad = 4k^2(2k - 5m) - 20km(2k - 5m) + 25m^2(2k - 5m)\\
&\quad = 8k^3 - 20k^2m - 40k^2m + 100km^2 + 50km^2 - 125m^3\\
&\quad = 8k^3 - 60k^2m + 150km^2 - 125m^3 \qquad \text{Combine like terms.}
\end{aligned}$$

TRY YOUR TURN 4

YOUR TURN 4 Find $(3x + 2y)^3$.

Notice in the first part of Example 5, when we multiplied $(2k - 5m)$ by itself, that the product of the square of a binomial is the square of the first term, $(2k)^2$, plus twice the product of the two terms, $(2)(2k)(-5m)$, plus the square of the last term, $(-5k)^2$.

CAUTION Avoid the common error of writing $(x + y)^2 = x^2 + y^2$. As the first step of Example 5 shows, the square of a binomial has three terms, so

$$(x + y)^2 = x^2 + 2xy + y^2.$$

Furthermore, higher powers of a binomial also result in more than two terms. For example, verify by multiplication that

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

Remember, for any value of $n \neq 1$,

$$(x + y)^n \neq x^n + y^n.$$

R.1 EXERCISES

Perform the indicated operations.

1. $(2x^2 - 6x + 11) + (-3x^2 + 7x - 2)$

2. $(-4y^2 - 3y + 8) - (2y^2 - 6y - 2)$

3. $-6(2q^2 + 4q - 3) + 4(-q^2 + 7q - 3)$

4. $2(3r^2 + 4r + 2) - 3(-r^2 + 4r - 5)$

5. $(0.613x^2 - 4.215x + 0.892) - 0.47(2x^2 - 3x + 5)$

6. $0.5(5r^2 + 3.2r - 6) - (1.7r^2 - 2r - 1.5)$

7. $-9m(2m^2 + 3m - 1)$

8. $6x(-2x^3 + 5x + 6)$

9. $(3t - 2y)(3t + 5y)$

10. $(9k + q)(2k - q)$

11. $(2 - 3x)(2 + 3x)$

12. $(6m + 5)(6m - 5)$

13. $\left(\frac{2}{5}y + \frac{1}{8}z\right)\left(\frac{3}{5}y + \frac{1}{2}z\right)$

14. $\left(\frac{3}{4}r - \frac{2}{3}s\right)\left(\frac{5}{4}r + \frac{1}{3}s\right)$

15. $(3p - 1)(9p^2 + 3p + 1)$

16. $(3p + 2)(5p^2 + p - 4)$

17. $(2m + 1)(4m^2 - 2m + 1)$

18. $(k + 2)(12k^3 - 3k^2 + k + 1)$

19. $(x + y + z)(3x - 2y - z)$

20. $(r + 2s - 3t)(2r - 2s + t)$

21. $(x + 1)(x + 2)(x + 3)$

22. $(x - 1)(x + 2)(x - 3)$

23. $(x + 2)^2$

24. $(2a - 4b)^2$

25. $(x - 2y)^3$

26. $(3x + y)^3$

YOUR TURN ANSWERS

1. $-9x^2 + 8x + 13$

2. $12y^3 + 2y^2 - 19y - 10$

3. $6x^2 + 19x - 7$

4. $27x^3 + 54x^2y + 36xy^2 + 8y^3$

R.2 Factoring

Multiplication of polynomials relies on the distributive property. The reverse process, where a polynomial is written as a product of other polynomials, is called **factoring**. For example, one way to factor the number 18 is to write it as the product $9 \cdot 2$; both 9 and 2 are **factors** of 18. Usually, only integers are used as factors of integers. The number 18 can also be written with three integer factors as $2 \cdot 3 \cdot 3$.

The Greatest Common Factor

To factor the algebraic expression $15m + 45$, first note that both $15m$ and 45 are divisible by 15; $15m = 15 \cdot m$ and $45 = 15 \cdot 3$. By the distributive property,

$$15m + 45 = 15 \cdot m + 15 \cdot 3 = 15(m + 3).$$

Both 15 and $m + 3$ are factors of $15m + 45$. Since 15 divides into both terms of $15m + 45$ (and is the largest number that will do so), 15 is the **greatest common factor** for the polynomial $15m + 45$. The process of writing $15m + 45$ as $15(m + 3)$ is often called **factoring out** the greatest common factor.

EXAMPLE 1 Factoring

Factor out the greatest common factor.

(a) $12p - 18q$

SOLUTION Both $12p$ and $18q$ are divisible by 6. Therefore,

$$12p - 18q = 6 \cdot 2p - 6 \cdot 3q = 6(2p - 3q).$$

(b) $8x^3 - 9x^2 + 15x$

SOLUTION Each of these terms is divisible by x.

$$\begin{aligned} 8x^3 - 9x^2 + 15x &= (8x^2) \cdot x - (9x) \cdot x + 15 \cdot x \\ &= x(8x^2 - 9x + 15) \quad \text{or} \quad (8x^2 - 9x + 15)x \end{aligned}$$

TRY YOUR TURN 1

YOUR TURN 1 Factor $4z^4 + 4z^3 + 18z^2$.

One can always check factorization by finding the product of the factors and comparing it to the original expression.

CAUTION When factoring out the greatest common factor in an expression like $2x^2 + x$, be careful to remember the 1 in the second term.

$$2x^2 + x = 2x^2 + 1x = x(2x + 1), \qquad \text{not } x(2x).$$

Factoring Trinomials

A polynomial that has no greatest common factor (other than 1) may still be factorable. For example, the polynomial $x^2 + 5x + 6$ can be factored as $(x + 2)(x + 3)$. To see that this is correct, find the product $(x + 2)(x + 3)$; you should get $x^2 + 5x + 6$. A polynomial such as this with three terms is called a **trinomial**. To factor a trinomial of the form $x^2 + bx + c$, where the coefficient of x^2 is 1, use FOIL backwards. We look for two factors of c whose sum is b. When c is positive, its factors must have the same sign. Since b is the sum of these two factors, the factors will have the same sign as b. When c is negative, its factors have opposite signs. Again, since b is the sum of these two factors, the factor with the greater absolute value will have the same sign as b.

EXAMPLE 2 Factoring a Trinomial

Factor $y^2 + 8y + 15$.

SOLUTION Since the coefficient of y^2 is 1, factor by finding two numbers whose *product* is 15 and whose *sum* is 8. Because the constant and the middle term are positive, the numbers must both be positive. Begin by listing all pairs of positive integers having a product of 15. As you do this, also form the sum of each pair of numbers.

Products	**Sums**
$15 \cdot 1 = 15$	$15 + 1 = 16$
$5 \cdot 3 = 15$	$5 + 3 = 8$

The numbers 5 and 3 have a product of 15 and a sum of 8. Thus, $y^2 + 8y + 15$ factors as

$$y^2 + 8y + 15 = (y + 5)(y + 3).$$

The answer can also be written as $(y + 3)(y + 5)$.

TRY YOUR TURN 2

YOUR TURN 2 Factor $x^2 - 3x - 10$.

If the coefficient of the squared term is *not* 1, work as shown on the next page.

EXAMPLE 3 Factoring a Trinomial

Factor $4x^2 + 8xy - 5y^2$.

SOLUTION The possible factors of $4x^2$ are $4x$ and x or $2x$ and $2x$; the possible factors of $-5y^2$ are $-5y$ and y or $5y$ and $-y$. Try various combinations of these factors until one works (if, indeed, any work). For example, try the product $(x + 5y)(4x - y)$.

$$\begin{aligned}(x + 5y)(4x - y) &= 4x^2 - xy + 20xy - 5y^2 \\ &= 4x^2 + 19xy - 5y^2\end{aligned}$$

This product is not correct, so try another combination.

$$\begin{aligned}(2x - y)(2x + 5y) &= 4x^2 + 10xy - 2xy - 5y^2 \\ &= 4x^2 + 8xy - 5y^2\end{aligned}$$

Since this combination gives the correct polynomial,

$$4x^2 + 8xy - 5y^2 = (2x - y)(2x + 5y).$$

TRY YOUR TURN 3

YOUR TURN 3 Factor $6a^2 + 5ab - 4b^2$.

Special Factorizations

Four special factorizations occur so often that they are listed here for future reference.

Special Factorizations

$x^2 - y^2 = (x + y)(x - y)$	**Difference of two squares**
$x^2 + 2xy + y^2 = (x + y)^2$	**Perfect square**
$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$	**Difference of two cubes**
$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$	**Sum of two cubes**

A polynomial that cannot be factored is called a **prime polynomial**.

EXAMPLE 4 Factoring Polynomials

Factor each polynomial, if possible.

(a) $64p^2 - 49q^2 = (8p)^2 - (7q)^2 = (8p + 7q)(8p - 7q)$ — Difference of two squares

(b) $x^2 + 36$ is a prime polynomial.

(c) $x^2 + 12x + 36 = x^2 + 2(x)(6) + 6^2 = (x + 6)^2$ — Perfect square

(d) $9y^2 - 24yz + 16z^2 = (3y)^2 + 2(3y)(-4z) + (-4z)^2$ — Perfect square
$= [3y + (-4z)]^2 = (3y - 4z)^2$

(e) $y^3 - 8 = y^3 - 2^3 = (y - 2)(y^2 + 2y + 4)$ — Difference of two cubes

(f) $m^3 + 125 = m^3 + 5^3 = (m + 5)(m^2 - 5m + 25)$ — Sum of two cubes

(g) $8k^3 - 27z^3 = (2k)^3 - (3z)^3 = (2k - 3z)(4k^2 + 6kz + 9z^2)$ — Difference of two cubes

(h) $p^4 - 1 = (p^2 + 1)(p^2 - 1) = (p^2 + 1)(p + 1)(p - 1)$ — Difference of two squares

CAUTION In factoring, always look for a common factor first. Since $36x^2 - 4y^2$ has a common factor of 4,

$$36x^2 - 4y^2 = 4(9x^2 - y^2) = 4(3x + y)(3x - y).$$

It would be incomplete to factor it as

$$36x^2 - 4y^2 = (6x + 2y)(6x - 2y),$$

since each factor can be factored still further. To *factor* means to factor completely, so that each polynomial factor is prime.

R.2 EXERCISES

Factor each polynomial. If a polynomial cannot be factored, write *prime*. Factor out the greatest common factor as necessary.

1. $7a^3 + 14a^2$

2. $3y^3 + 24y^2 + 9y$

3. $13p^4q^2 - 39p^3q + 26p^2q^2$

4. $60m^4 - 120m^3n + 50m^2n^2$

5. $m^2 - 5m - 14$

6. $x^2 + 4x - 5$

7. $z^2 + 9z + 20$

8. $b^2 - 8b + 7$

9. $a^2 - 6ab + 5b^2$

10. $s^2 + 2st - 35t^2$

11. $y^2 - 4yz - 21z^2$

12. $3x^2 + 4x - 7$

13. $3a^2 + 10a + 7$

14. $15y^2 + y - 2$

15. $21m^2 + 13mn + 2n^2$

16. $6a^2 - 48a - 120$

17. $3m^3 + 12m^2 + 9m$

18. $4a^2 + 10a + 6$

19. $24a^4 + 10a^3b - 4a^2b^2$

20. $24x^4 + 36x^3y - 60x^2y^2$

21. $x^2 - 64$

22. $9m^2 - 25$

23. $10x^2 - 160$

24. $9x^2 + 64$

25. $z^2 + 14zy + 49y^2$

26. $s^2 - 10st + 25t^2$

27. $9p^2 - 24p + 16$

28. $a^3 - 216$

29. $27r^3 - 64s^3$

30. $3m^3 + 375$

31. $x^4 - y^4$

32. $16a^4 - 81b^4$

YOUR TURN ANSWERS

1. $2z^2(2z^2 + 2z + 9)$

2. $(x + 2)(x - 5)$

3. $(2a - b)(3a + 4b)$

R.3 Rational Expressions

Many algebraic fractions are **rational expressions**, which are quotients of polynomials with nonzero denominators. Examples include

$$\frac{8}{x-1}, \quad \frac{3x^2+4x}{5x-6}, \quad \text{and} \quad \frac{2y+1}{y^2}.$$

Next, we summarize properties for working with rational expressions.

Properties of Rational Expressions

For all mathematical expressions P, Q, R, and S, with $Q \neq 0$ and $S \neq 0$:

$\frac{P}{Q} = \frac{PS}{QS}$	**Fundamental property**
$\frac{P}{Q} + \frac{R}{Q} = \frac{P+R}{Q}$	**Addition**
$\frac{P}{Q} - \frac{R}{Q} = \frac{P-R}{Q}$	**Subtraction**
$\frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS}$	**Multiplication**
$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R} \quad (R \neq 0)$	**Division**

When writing a rational expression in lowest terms, we may need to use the fact that $\frac{a^m}{a^n} = a^{m-n}$. For example,

$$\frac{x^4}{3x} = \frac{1x^4}{3x} = \frac{1}{3} \cdot \frac{x^4}{x} = \frac{1}{3} \cdot x^{4-1} = \frac{1}{3}x^3 = \frac{x^3}{3}.$$

EXAMPLE 1 Reducing Rational Expressions

Write each rational expression in lowest terms, that is, reduce the expression as much as possible.

(a) $\dfrac{8x+16}{4} = \dfrac{8(x+2)}{4} = \dfrac{4\cdot 2(x+2)}{4} = 2(x+2)$

Factor both the numerator and denominator in order to identify any common factors, which have a quotient of 1. The answer could also be written as $2x+4$.

(b) $\dfrac{k^2+7k+12}{k^2+2k-3} = \dfrac{(k+4)(k+3)}{(k-1)(k+3)} = \dfrac{k+4}{k-1}$

The answer cannot be further reduced.

TRY YOUR TURN 1

YOUR TURN 1 Write in lowest terms $\dfrac{z^2+5z+6}{2z^2+7z+3}$.

CAUTION One of the most common errors in algebra involves incorrect use of the fundamental property of rational expressions. Only common *factors* may be divided or "canceled." It is essential to factor rational expressions before writing them in lowest terms. In Example 1(b), for instance, it is not correct to "cancel" k^2 (or cancel k, or divide 12 by -3) because the additions and subtraction must be performed first. Here they cannot be performed, so it is not possible to divide. After factoring, however, the fundamental property can be used to write the expression in lowest terms.

EXAMPLE 2 Combining Rational Expressions

Perform each operation.

(a) $\dfrac{3y+9}{6}\cdot\dfrac{18}{5y+15}$

SOLUTION Factor where possible, then multiply numerators and denominators and reduce to lowest terms.

$$\begin{aligned}\frac{3y+9}{6}\cdot\frac{18}{5y+15} &= \frac{3(y+3)}{6}\cdot\frac{18}{5(y+3)} && \text{Factor.}\\ &= \frac{3\cdot 18(y+3)}{6\cdot 5(y+3)} && \text{Multiply.}\\ &= \frac{3\cdot\cancel{6}\cdot 3\cancel{(y+3)}}{\cancel{6}\cdot 5\cancel{(y+3)}} = \frac{3\cdot 3}{5} = \frac{9}{5} && \text{Reduce to lowest terms.}\end{aligned}$$

(b) $\dfrac{m^2+5m+6}{m+3}\cdot\dfrac{m}{m^2+3m+2}$

SOLUTION Factor where possible.

$$\begin{aligned}&\frac{(m+2)(m+3)}{m+3}\cdot\frac{m}{(m+2)(m+1)} && \text{Factor.}\\ &= \frac{m\cancel{(m+2)}\cancel{(m+3)}}{\cancel{(m+3)}\cancel{(m+2)}(m+1)} = \frac{m}{m+1} && \text{Reduce to lowest terms.}\end{aligned}$$

(c) $\dfrac{9p-36}{12}\div\dfrac{5(p-4)}{18}$

SOLUTION Use the division property of rational expressions.

$$\begin{aligned}\frac{9p-36}{12}\div\frac{5(p-4)}{18} &= \frac{9p-36}{12}\cdot\frac{18}{5(p-4)} && \text{Invert and multiply.}\\ &= \frac{9\cancel{(p-4)}}{\cancel{6}\cdot 2}\cdot\frac{\cancel{6}\cdot 3}{5\cancel{(p-4)}} = \frac{27}{10} && \text{Factor and reduce to lowest terms.}\end{aligned}$$

(d) $\dfrac{4}{5k} - \dfrac{11}{5k}$

SOLUTION As shown in the list of properties, to subtract two rational expressions that have the same denominators, subtract the numerators while keeping the same denominator.

$$\frac{4}{5k} - \frac{11}{5k} = \frac{4 - 11}{5k} = -\frac{7}{5k}$$

(e) $\dfrac{7}{p} + \dfrac{9}{2p} + \dfrac{1}{3p}$

SOLUTION These three fractions cannot be added until their denominators are the same. A **common denominator** into which p, $2p$, and $3p$ all divide is $6p$. Note that $12p$ is also a common denominator, but $6p$ is the **least common denominator**. Use the fundamental property to rewrite each rational expression with a denominator of $6p$.

$$\begin{aligned}
\frac{7}{p} + \frac{9}{2p} + \frac{1}{3p} &= \frac{6 \cdot 7}{6 \cdot p} + \frac{3 \cdot 9}{3 \cdot 2p} + \frac{2 \cdot 1}{2 \cdot 3p} && \text{Rewrite with common denominator } 6p. \\
&= \frac{42}{6p} + \frac{27}{6p} + \frac{2}{6p} \\
&= \frac{42 + 27 + 2}{6p} \\
&= \frac{71}{6p}
\end{aligned}$$

(f) $\dfrac{x + 1}{x^2 + 5x + 6} - \dfrac{5x - 1}{x^2 - x - 12}$

SOLUTION To find the least common denominator, we first factor each denominator. Then we change each fraction so they all have the same denominator, being careful to multiply only by quotients that equal 1.

$$\begin{aligned}
&\frac{x + 1}{x^2 + 5x + 6} - \frac{5x - 1}{x^2 - x - 12} \\
&= \frac{x + 1}{(x + 2)(x + 3)} - \frac{5x - 1}{(x + 3)(x - 4)} && \text{Factor denominators.} \\
&= \frac{x + 1}{(x + 2)(x + 3)} \cdot \frac{(x - 4)}{(x - 4)} - \frac{5x - 1}{(x + 3)(x - 4)} \cdot \frac{(x + 2)}{(x + 2)} && \text{Rewrite with common denominators.} \\
&= \frac{(x^2 - 3x - 4) - (5x^2 + 9x - 2)}{(x + 2)(x + 3)(x - 4)} && \text{Multiply numerators.} \\
&= \frac{-4x^2 - 12x - 2}{(x + 2)(x + 3)(x - 4)} && \text{Subtract.} \\
&= \frac{-2(2x^2 + 6x + 1)}{(x + 2)(x + 3)(x - 4)} && \text{Factor numerator.}
\end{aligned}$$

Because the numerator cannot be factored further, we leave our answer in this form. We could also multiply out the denominator, but factored form is usually more useful.

TRY YOUR TURN 2

YOUR TURN 2 Perform each of the following operations.

(a) $\dfrac{z^2 + 5z + 6}{2z^2 - 5z - 3} \cdot \dfrac{2z^2 - z - 1}{z^2 + 2z - 3}$.

(b) $\dfrac{a - 3}{a^2 + 3a + 2} + \dfrac{5a}{a^2 - 4}$.

R.3 EXERCISES

Write each rational expression in lowest terms.

1. $\dfrac{5v^2}{35v}$
2. $\dfrac{25p^3}{10p^2}$
3. $\dfrac{8k+16}{9k+18}$
4. $\dfrac{2(t-15)}{(t-15)(t+2)}$
5. $\dfrac{4x^3-8x^2}{4x^2}$
6. $\dfrac{36y^2+72y}{9y}$
7. $\dfrac{m^2-4m+4}{m^2+m-6}$
8. $\dfrac{r^2-r-6}{r^2+r-12}$
9. $\dfrac{3x^2+3x-6}{x^2-4}$
10. $\dfrac{z^2-5z+6}{z^2-4}$
11. $\dfrac{m^4-16}{4m^2-16}$
12. $\dfrac{6y^2+11y+4}{3y^2+7y+4}$

Perform the indicated operations.

13. $\dfrac{9k^2}{25}\cdot\dfrac{5}{3k}$
14. $\dfrac{15p^3}{9p^2}\div\dfrac{6p}{10p^2}$
15. $\dfrac{3a+3b}{4c}\cdot\dfrac{12}{5(a+b)}$
16. $\dfrac{a-3}{16}\div\dfrac{a-3}{32}$
17. $\dfrac{2k-16}{6}\div\dfrac{4k-32}{3}$
18. $\dfrac{9y-18}{6y+12}\cdot\dfrac{3y+6}{15y-30}$
19. $\dfrac{4a+12}{2a-10}\div\dfrac{a^2-9}{a^2-a-20}$
20. $\dfrac{6r-18}{9r^2+6r-24}\cdot\dfrac{12r-16}{4r-12}$
21. $\dfrac{k^2+4k-12}{k^2+10k+24}\cdot\dfrac{k^2+k-12}{k^2-9}$
22. $\dfrac{m^2+3m+2}{m^2+5m+4}\div\dfrac{m^2+5m+6}{m^2+10m+24}$
23. $\dfrac{2m^2-5m-12}{m^2-10m+24}\div\dfrac{4m^2-9}{m^2-9m+18}$
24. $\dfrac{4n^2+4n-3}{6n^2-n-15}\cdot\dfrac{8n^2+32n+30}{4n^2+16n+15}$
25. $\dfrac{a+1}{2}-\dfrac{a-1}{2}$
26. $\dfrac{3}{p}+\dfrac{1}{2}$
27. $\dfrac{6}{5y}-\dfrac{3}{2}$
28. $\dfrac{1}{6m}+\dfrac{2}{5m}+\dfrac{4}{m}$
29. $\dfrac{1}{m-1}+\dfrac{2}{m}$
30. $\dfrac{5}{2r+3}-\dfrac{2}{r}$
31. $\dfrac{8}{3(a-1)}+\dfrac{2}{a-1}$
32. $\dfrac{2}{5(k-2)}+\dfrac{3}{4(k-2)}$
33. $\dfrac{4}{x^2+4x+3}+\dfrac{3}{x^2-x-2}$
34. $\dfrac{y}{y^2+2y-3}-\dfrac{1}{y^2+4y+3}$
35. $\dfrac{3k}{2k^2+3k-2}-\dfrac{2k}{2k^2-7k+3}$
36. $\dfrac{4m}{3m^2+7m-6}-\dfrac{m}{3m^2-14m+8}$
37. $\dfrac{2}{a+2}+\dfrac{1}{a}+\dfrac{a-1}{a^2+2a}$
38. $\dfrac{5x+2}{x^2-1}+\dfrac{3}{x^2+x}-\dfrac{1}{x^2-x}$

YOUR TURN ANSWERS

1. $(z+2)/(2z+1)$
2. **(a)** $(z+2)/(z-3)$
   (b) $6(a^2+1)/[(a-2)(a+2)(a+1)]$

R.4 Equations

Linear Equations

Equations that can be written in the form $ax+b=0$, where a and b are real numbers, with $a \neq 0$, are **linear equations**. Examples of linear equations include $5y+9=16$, $8x=4$, and $-3p+5=-8$. Equations that are *not* linear include absolute value equations such as $|x|=4$. The following properties are used to solve linear equations.

Properties of Equality

For all real numbers a, b, and c:

1. **If $a=b$, then $a+c=b+c$.** (The same number may be added to both sides of an equation.) — **Addition property of equality**
2. **If $a=b$, then $ac=bc$.** (Both sides of an equation may be multiplied by the same number.) — **Multiplication property of equality**

EXAMPLE 1 Solving Linear Equations

Solve the following equations.

(a) $x - 2 = 3$

SOLUTION The goal is to isolate the variable. Using the addition property of equality yields

$$x - 2 + 2 = 3 + 2, \qquad \text{or} \qquad x = 5.$$

(b) $\dfrac{x}{2} = 3$

SOLUTION Using the multiplication property of equality yields

$$2 \cdot \frac{x}{2} = 2 \cdot 3, \qquad \text{or} \qquad x = 6.$$

The following example shows how these properties are used to solve linear equations. The goal is to isolate the variable. The solutions should always be checked by substitution into the original equation.

EXAMPLE 2 Solving a Linear Equation

Solve $2x - 5 + 8 = 3x + 2(2 - 3x)$.

SOLUTION

$2x - 5 + 8 = 3x + 4 - 6x$	Distributive property
$2x + 3 = -3x + 4$	Combine like terms.
$5x + 3 = 4$	Add $3x$ to both sides.
$5x = 1$	Add -3 to both sides.
$x = \dfrac{1}{5}$	Multiply both sides by $\frac{1}{5}$.

YOUR TURN 1 Solve $3x - 7 = 4(5x + 2) - 7x$.

Check by substituting into the original equation. The left side becomes $2(1/5) - 5 + 8$ and the right side becomes $3(1/5) + 2[2 - 3(1/5)]$. Verify that both of these expressions simplify to $17/5$.

TRY YOUR TURN 1

Quadratic Equations

An equation with 2 as the greatest exponent of the variable is a *quadratic equation*. A **quadratic equation** has the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$. A quadratic equation written in the form $ax^2 + bx + c = 0$ is said to be in **standard form**.

The simplest way to solve a quadratic equation, but one that is not always applicable, is by factoring. This method depends on the **zero-factor property**.

Zero-Factor Property

If a and b are real numbers, with $ab = 0$, then either

$$a = 0 \text{ or } b = 0 \quad \text{(or both)}.$$

EXAMPLE 3 Solving a Quadratic Equation

Solve $6r^2 + 7r = 3$.

SOLUTION First write the equation in standard form.

$$6r^2 + 7r - 3 = 0$$

Now factor $6r^2 + 7r - 3$ to get

$$(3r - 1)(2r + 3) = 0.$$

By the zero-factor property, the product $(3r - 1)(2r + 3)$ can equal 0 if and only if

$$3r - 1 = 0 \quad \text{or} \quad 2r + 3 = 0.$$

Solve each of these equations separately to find that the solutions are $1/3$ and $-3/2$. Check these solutions by substituting them into the original equation. **TRY YOUR TURN 2**

YOUR TURN 2 Solve $2m^2 + 7m = 15$.

CAUTION Remember, the zero-factor property requires that the product of two (or more) factors be equal to *zero*, not some other quantity. It would be incorrect to use the zero-factor property with an equation in the form $(x + 3)(x - 1) = 4$, for example.

If a quadratic equation cannot be solved easily by factoring, use the *quadratic formula*. (The derivation of the quadratic formula is given in most algebra books.)

Quadratic Formula

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

EXAMPLE 4 Quadratic Formula

Solve $x^2 - 4x - 5 = 0$ by the quadratic formula.

SOLUTION The equation is already in standard form (it has 0 alone on one side of the equal sign), so the values of a, b, and c from the quadratic formula are easily identified. The coefficient of the squared term gives the value of a; here, $a = 1$. Also, $b = -4$ and $c = -5$, where b is the coefficient of the linear term and c is the constant coefficient. (Be careful to use the correct signs.) Substitute these values into the quadratic formula.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-5)}}{2(1)} \qquad a = 1, b = -4, c = -5$$

$$x = \frac{4 \pm \sqrt{16 + 20}}{2} \qquad (-4)^2 = (-4)(-4) = 16$$

$$x = \frac{4 \pm 6}{2} \qquad \sqrt{16 + 20} = \sqrt{36} = 6$$

The $\pm$ sign represents the two solutions of the equation. To find both of the solutions, first use $+$ and then use $-$.

$$x = \frac{4 + 6}{2} = \frac{10}{2} = 5 \quad \text{or} \quad x = \frac{4 - 6}{2} = \frac{-2}{2} = -1$$

The two solutions are 5 and -1.

CAUTION Notice in the quadratic formula that the square root is added to or subtracted from the value of $-b$ *before* dividing by $2a$.

EXAMPLE 5 Quadratic Formula

Solve $x^2 + 1 = 4x$.

SOLUTION First, add $-4x$ on both sides of the equal sign in order to get the equation in standard form.

$$x^2 - 4x + 1 = 0$$

Now identify the values of a, b, and c. Here $a = 1$, $b = -4$, and $c = 1$. Substitute these numbers into the quadratic formula.

$$\begin{aligned} x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} && a = 1, b = -4, c = 1 \\ &= \frac{4 \pm \sqrt{16 - 4}}{2} \\ &= \frac{4 \pm \sqrt{12}}{2} \end{aligned}$$

Simplify the solutions by writing $\sqrt{12}$ as $\sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$. Substituting $2\sqrt{3}$ for $\sqrt{12}$ gives

$$\begin{aligned} x &= \frac{4 \pm 2\sqrt{3}}{2} \\ &= \frac{2(2 \pm \sqrt{3})}{2} && \text{Factor } 4 \pm 2\sqrt{3}. \\ &= 2 \pm \sqrt{3}. && \text{Reduce to lowest terms.} \end{aligned}$$

The two solutions are $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

The exact values of the solutions are $2 + \sqrt{3}$ and $2 - \sqrt{3}$. The $\sqrt{\ }$ key on a calculator gives decimal approximations of these solutions (to the nearest thousandth):

YOUR TURN 3 Solve $z^2 + 6 = 8z$.

$$2 + \sqrt{3} \approx 2 + 1.732 = 3.732^*$$
$$2 - \sqrt{3} \approx 2 - 1.732 = 0.268$$

TRY YOUR TURN 3

NOTE Sometimes the quadratic formula will give a result with a negative number under the radical sign, such as $3 \pm \sqrt{-5}$. A solution of this type is a complex number. Since this text deals only with real numbers, such solutions cannot be used.

Equations with Fractions

When an equation includes fractions, first eliminate all denominators by multiplying both sides of the equation by a common denominator, a number that can be divided (with no remainder) by each denominator in the equation. When an equation involves fractions with variable denominators, it is *necessary* to check all solutions in the original equation to be sure that no solution will lead to a zero denominator.

EXAMPLE 6 Solving Rational Equations

Solve each equation.

(a) $\dfrac{r}{10} - \dfrac{2}{15} = \dfrac{3r}{20} - \dfrac{1}{5}$

SOLUTION The denominators are 10, 15, 20, and 5. Each of these numbers can be divided into 60, so 60 is a common denominator. Multiply both sides of the equation by

*The symbol $\approx$ means "is approximately equal to."

60 and use the distributive property. (If a common denominator cannot be found easily, all the denominators in the problem can be multiplied together to produce one.)

$$\frac{r}{10} - \frac{2}{15} = \frac{3r}{20} - \frac{1}{5}$$

$$60\left(\frac{r}{10} - \frac{2}{15}\right) = 60\left(\frac{3r}{20} - \frac{1}{5}\right) \qquad \text{Multiply by the common denominator.}$$

$$60\left(\frac{r}{10}\right) - 60\left(\frac{2}{15}\right) = 60\left(\frac{3r}{20}\right) - 60\left(\frac{1}{5}\right) \qquad \text{Distributive property}$$

$$6r - 8 = 9r - 12$$

Add $-9r$ and 8 to both sides.

$$6r - 8 + (-9r) + 8 = 9r - 12 + (-9r) + 8$$

$$-3r = -4$$

$$r = \frac{4}{3} \qquad \text{Multiply each side by } -\tfrac{1}{3}.$$

Check by substituting into the original equation.

(b) $\frac{3}{x^2} - 12 = 0$

SOLUTION Begin by multiplying both sides of the equation by x^2 to get $3 - 12x^2 = 0$. This equation could be solved by using the quadratic formula with $a = -12$, $b = 0$, and $c = 3$. Another method that works well for the type of quadratic equation in which $b = 0$ is shown below.

$$3 - 12x^2 = 0$$

$$3 = 12x^2 \qquad \text{Add } 12x^2.$$

$$\frac{1}{4} = x^2 \qquad \text{Multiply by } \tfrac{1}{12}.$$

$$\pm\frac{1}{2} = x \qquad \text{Take square roots.}$$

Verify that there are two solutions, $-1/2$ and $1/2$.

(c) $\frac{2}{k} - \frac{3k}{k+2} = \frac{k}{k^2 + 2k}$

SOLUTION Factor $k^2 + 2k$ as $k(k + 2)$. The least common denominator for all the fractions is $k(k + 2)$. Multiplying both sides by $k(k + 2)$ gives the following:

$$k(k + 2) \cdot \left(\frac{2}{k} - \frac{3k}{k+2}\right) = k(k + 2) \cdot \frac{k}{k^2 + 2k}$$

$$2(k + 2) - 3k(k) = k$$

$$2k + 4 - 3k^2 = k \qquad \text{Distributive property}$$

$$-3k^2 + k + 4 = 0 \qquad \text{Add } -k\text{; rearrange terms.}$$

$$3k^2 - k - 4 = 0 \qquad \text{Multiply by } -1.$$

$$(3k - 4)(k + 1) = 0 \qquad \text{Factor.}$$

$$3k - 4 = 0 \quad \text{or} \quad k + 1 = 0$$

$$k = \frac{4}{3} \qquad k = -1$$

Verify that the solutions are $4/3$ and -1.

TRY YOUR TURN 4

YOUR TURN 4 Solve $\frac{1}{x^2 - 4} + \frac{2}{x - 2} = \frac{1}{x}$.

CAUTION It is possible to get, as a solution of a rational equation, a number that makes one or more of the denominators in the original equation equal to zero. That number is not a solution, so it is *necessary* to check all potential solutions of rational equations. These introduced solutions are called **extraneous solutions**.

EXAMPLE 7 Solving a Rational Equation

Solve $\dfrac{2}{x-3} + \dfrac{1}{x} = \dfrac{6}{x(x-3)}$.

SOLUTION The common denominator is $x(x-3)$. Multiply both sides by $x(x-3)$ and solve the resulting equation.

$$x(x-3)\cdot\left(\frac{2}{x-3} + \frac{1}{x}\right) = x(x-3)\cdot\left[\frac{6}{x(x-3)}\right]$$
$$2x + x - 3 = 6$$
$$3x = 9$$
$$x = 3$$

Checking this potential solution by substitution into the original equation shows that 3 makes two denominators 0. Thus, 3 cannot be a solution, so there is no solution for this equation.

R.4 EXERCISES

Solve each equation.

1. $2x + 8 = x - 4$

2. $5x + 2 = 8 - 3x$

3. $0.2m - 0.5 = 0.1m + 0.7$

4. $\frac{2}{3}k - k + \frac{3}{8} = \frac{1}{2}$

5. $3r + 2 - 5(r + 1) = 6r + 4$

6. $5(a + 3) + 4a - 5 = -(2a - 4)$

7. $2[3m - 2(3 - m) - 4] = 6m - 4$

8. $4[2p - (3 - p) + 5] = -7p - 2$

Solve each equation by factoring or by using the quadratic formula. If the solutions involve square roots, give both the exact solutions and the approximate solutions to three decimal places.

9. $x^2 + 5x + 6 = 0$

10. $x^2 = 3 + 2x$

11. $m^2 = 14m - 49$

12. $2k^2 - k = 10$

13. $12x^2 - 5x = 2$

14. $m(m - 7) = -10$

15. $4x^2 - 36 = 0$

16. $z(2z + 7) = 4$

17. $12y^2 - 48y = 0$

18. $3x^2 - 5x + 1 = 0$

19. $2m^2 - 4m = 3$

20. $p^2 + p - 1 = 0$

21. $k^2 - 10k = -20$

22. $5x^2 - 8x + 2 = 0$

23. $2r^2 - 7r + 5 = 0$

24. $2x^2 - 7x + 30 = 0$

25. $3k^2 + k = 6$

26. $5m^2 + 5m = 0$

Solve each equation.

27. $\dfrac{3x-2}{7} = \dfrac{x+2}{5}$

28. $\dfrac{x}{3} - 7 = 6 - \dfrac{3x}{4}$

29. $\dfrac{4}{x-3} - \dfrac{8}{2x+5} + \dfrac{3}{x-3} = 0$

30. $\dfrac{5}{p-2} - \dfrac{7}{p+2} = \dfrac{12}{p^2-4}$

31. $\dfrac{2m}{m-2} - \dfrac{6}{m} = \dfrac{12}{m^2-2m}$

32. $\dfrac{2y}{y-1} = \dfrac{5}{y} + \dfrac{10-8y}{y^2-y}$

33. $\dfrac{1}{x-2} - \dfrac{3x}{x-1} = \dfrac{2x+1}{x^2-3x+2}$

34. $\dfrac{5}{a} + \dfrac{-7}{a+1} = \dfrac{a^2-2a+4}{a^2+a}$

35. $\dfrac{5}{b+5} - \dfrac{4}{b^2+2b} = \dfrac{6}{b^2+7b+10}$

36. $\dfrac{2}{x^2-2x-3} + \dfrac{5}{x^2-x-6} = \dfrac{1}{x^2+3x+2}$

37. $\dfrac{4}{2x^2+3x-9} + \dfrac{2}{2x^2-x-3} = \dfrac{3}{x^2+4x+3}$

YOUR TURN ANSWERS

1. $-3/2$

2. $3/2, -5$

3. $4 \pm \sqrt{10}$

4. $-1, -4$

R.5 Inequalities

To write that one number is greater than or less than another number, we use the following symbols.

Inequality Symbols

$<$ means *is less than* $\qquad$ $\leq$ means *is less than or equal to*

$>$ means *is greater than* $\qquad$ $\geq$ means *is greater than or equal to*

Linear Inequalities

An equation states that two expressions are equal; an **inequality** states that they are unequal. A **linear inequality** is an inequality that can be simplified to the form $ax < b$. (Properties introduced in this section are given only for $<$, but they are equally valid for $>$, $\leq$, or $\geq$.) Linear inequalities are solved with the following properties.

Properties of Inequality

For all real numbers a, b, and c:

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$ and if $c > 0$, then $ac < bc$.
3. If $a < b$ and if $c < 0$, then $ac < bc$.

Pay careful attention to property 3; it says that if both sides of an inequality are multiplied by a negative number, the direction of the inequality symbol must be reversed.

EXAMPLE 1 Solving a Linear Inequality

Solve $4 - 3y \leq 7 + 2y$.

SOLUTION Use the properties of inequality.

$$4 - 3y + (-4) \leq 7 + 2y + (-4) \qquad \text{Add } -4 \text{ to both sides.}$$

$$-3y \leq 3 + 2y$$

Remember that *adding* the same number to both sides never changes the direction of the inequality symbol.

$$-3y + (-2y) \leq 3 + 2y + (-2y) \qquad \text{Add } -2y \text{ to both sides.}$$

$$-5y \leq 3$$

Multiply both sides by $-1/5$. Since $-1/5$ is negative, change the direction of the inequality symbol.

$$-\frac{1}{5}(-5y) \geq -\frac{1}{5}(3)$$

$$y \geq -\frac{3}{5}$$

TRY YOUR TURN 1

YOUR TURN 1 Solve $3z - 2 > 5z + 7$.

CAUTION It is a common error to forget to reverse the direction of the inequality sign when multiplying or dividing by a negative number. For example, to solve $-4x \leq 12$, we must multiply by $-1/4$ on both sides *and* reverse the inequality symbol to get $x \geq -3$.

The solution $y \geq -3/5$ in Example 1 represents an interval on the number line. **Interval notation** often is used for writing intervals. With interval notation, $y \geq -3/5$ is written as $[-3/5, \infty)$. This is an example of a **half-open interval**, since one endpoint, $-3/5$, is included. The **open interval** $(2, 5)$ corresponds to $2 < x < 5$, with neither endpoint included. The **closed interval** $[2, 5]$ includes both endpoints and corresponds to $2 \leq x \leq 5$.

The **graph** of an interval shows all points on a number line that correspond to the numbers in the interval. To graph the interval $[-3/5, \infty)$, for example, use a solid circle at $-3/5$, since $-3/5$ is part of the solution. To show that the solution includes all real numbers greater than or equal to $-3/5$, draw a heavy arrow pointing to the right (the positive direction). See Figure 1.

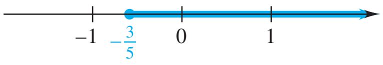


FIGURE 1

EXAMPLE 2 Graphing a Linear Inequality

Solve $-2 < 5 + 3m < 20$. Graph the solution.

SOLUTION The inequality $-2 < 5 + 3m < 20$ says that $5 + 3m$ is *between* -2 and 20. Solve this inequality with an extension of the properties given above. Work as follows, first adding -5 to each part.

$$-2 + (-5) < 5 + 3m + (-5) < 20 + (-5)$$
$$-7 < 3m < 15$$

Now multiply each part by $1/3$.

$$-\frac{7}{3} < m < 5$$

A graph of the solution is given in Figure 2; here open circles are used to show that $-7/3$ and 5 are *not* part of the graph.*

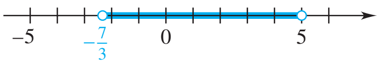


FIGURE 2

Quadratic Inequalities

A **quadratic inequality** has the form $ax^2 + bx + c > 0$ (or $<$, or $\leq$, or $\geq$). The greatest exponent is 2. The next few examples show how to solve quadratic inequalities.

EXAMPLE 3 Solving a Quadratic Inequality

Solve the quadratic inequality $x^2 - x < 12$.

SOLUTION Write the inequality with 0 on one side, as $x^2 - x - 12 < 0$. This inequality is solved with values of x that make $x^2 - x - 12$ negative (< 0). The quantity $x^2 - x - 12$ changes from positive to negative or from negative to positive at the points where it equals 0. For this reason, first solve the *equation* $x^2 - x - 12 = 0$.

$$x^2 - x - 12 = 0$$
$$(x - 4)(x + 3) = 0$$
$$x = 4 \quad \text{or} \quad x = -3$$

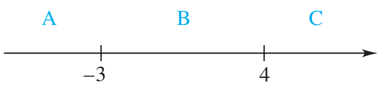


FIGURE 3

Locating -3 and 4 on a number line, as shown in Figure 3, determines three intervals A, B, and C. Decide which intervals include numbers that make $x^2 - x - 12$ negative by substituting any number from each interval into the polynomial. For example,

$$\text{choose } -4 \text{ from interval A: } (-4)^2 - (-4) - 12 = 8 > 0;$$
$$\text{choose } 0 \text{ from interval B: } 0^2 - 0 - 12 = -12 < 0;$$
$$\text{choose } 5 \text{ from interval C: } 5^2 - 5 - 12 = 8 > 0.$$

Only numbers in interval B satisfy the given inequality, so the solution is $(-3, 4)$. A graph of this solution is shown in Figure 4.

TRY YOUR TURN 2

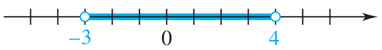


FIGURE 4

YOUR TURN 2 Solve $3y^2 \leq 16y + 12$.

*Some textbooks use brackets in place of solid circles for the graph of a closed interval, and parentheses in place of open circles for the graph of an open interval.

EXAMPLE 4 Solving a Polynomial Inequality

Solve the inequality $x^3 + 2x^2 - 3x \geq 0$.

SOLUTION This is not a quadratic inequality because of the x^3 term, but we solve it in a similar way by first factoring the polynomial.

$$
\begin{aligned}
x^3 + 2x^2 - 3x &= x(x^2 + 2x - 3) && \text{Factor out the common factor.} \\
&= x(x - 1)(x + 3) && \text{Factor the quadratic.}
\end{aligned}
$$

Now solve the corresponding equation.

$$x(x - 1)(x + 3) = 0$$

$$
\begin{array}{ccccc}
x = 0 & \text{or} & x - 1 = 0 & \text{or} & x + 3 = 0 \\
 & & x = 1 & & x = -3
\end{array}
$$

FIGURE 5

These three solutions determine four intervals on the number line: $(-\infty, -3)$, $(-3, 0)$, $(0, 1)$, and $(1, \infty)$. Substitute a number from each interval into the original inequality to determine that the solution consists of the numbers between -3 and 0 (including the endpoints) and all numbers that are greater than or equal to 1. See Figure 5. In interval notation, the solution is

$$[-3, 0] \cup [1, \infty).^*$$

Inequalities with Fractions

Inequalities with fractions are solved in a similar manner as quadratic inequalities.

EXAMPLE 5 Solving a Rational Inequality

Solve $\dfrac{2x - 3}{x} \geq 1$.

SOLUTION First solve the corresponding equation.

$$
\begin{aligned}
\frac{2x - 3}{x} &= 1 \\
2x - 3 &= x && \text{Multiply both sides by } x. \\
x &= 3 && \text{Solve for } x.
\end{aligned}
$$

The solution, $x = 3$, determines the intervals on the number line where the fraction may change from greater than 1 to less than 1. This change also may occur on either side of a number that makes the denominator equal 0. Here, the x-value that makes the denominator 0 is $x = 0$. Test each of the three intervals determined by the numbers 0 and 3.

$$
\begin{aligned}
&\text{For } (-\infty, 0), \text{ choose } -1: \frac{2(-1) - 3}{-1} = 5 \geq 1. \\
&\text{For } (0, 3), \quad \text{choose } \ 1: \frac{2(1) - 3}{1} = -1 \not\geq 1. \\
&\text{For } (3, \infty), \quad \text{choose } \ 4: \frac{2(4) - 3}{4} = \frac{5}{4} \geq 1.
\end{aligned}
$$

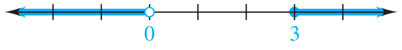


FIGURE 6

The symbol $\not\geq$ means "is *not* greater than or equal to." Testing the endpoints 0 and 3 shows that the solution is $(-\infty, 0) \cup [3, \infty)$, as shown in Figure 6.

CAUTION A common error is to try to solve the inequality in Example 5 by multiplying both sides by x. The reason this is wrong is that we don't know in the beginning whether x is positive, negative, or 0. If x is negative, the $\geq$ would change to $\leq$ according to the third property of inequality listed at the beginning of this section.

*The symbol $\cup$ indicates the *union* of two sets, which includes all elements in either set.

EXAMPLE 6 Solving a Rational Inequality

Solve $\dfrac{(x-1)(x+1)}{x} \le 0.$

SOLUTION We first solve the corresponding equation.

$$\frac{(x-1)(x+1)}{x} = 0$$

$$(x-1)(x+1) = 0 \qquad \text{Multiply both sides by } x.$$

$$x = 1 \quad \text{or} \quad x = -1 \qquad \text{Use the zero-factor property.}$$

Setting the denominator equal to 0 gives $x = 0$, so the intervals of interest are $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$. Testing a number from each region in the original inequality and checking the endpoints, we find the solution is

$$(-\infty, -1] \cup (0, 1],$$

FIGURE 7

as shown in Figure 7.

CAUTION Remember to solve the equation formed by setting the *denominator* equal to zero. Any number that makes the denominator zero always creates two intervals on the number line. For instance, in Example 6, substituting $x = 0$ makes the denominator of the rational inequality equal to 0, so we know that there may be a sign change from one side of 0 to the other (as was indeed the case).

EXAMPLE 7 Solving a Rational Inequality

Solve $\dfrac{x^2 - 3x}{x^2 - 9} < 4.$

SOLUTION Solve the corresponding equation.

$$\frac{x^2 - 3x}{x^2 - 9} = 4$$

$$x^2 - 3x = 4x^2 - 36 \qquad \text{Multiply by } x^2 - 9.$$

$$0 = 3x^2 + 3x - 36 \qquad \text{Get 0 on one side.}$$

$$0 = x^2 + x - 12 \qquad \text{Multiply by } \tfrac{1}{3}.$$

$$0 = (x+4)(x-3) \qquad \text{Factor.}$$

$$x = -4 \quad \text{or} \quad x = 3$$

Now set the denominator equal to 0 and solve that equation.

$$x^2 - 9 = 0$$

$$(x-3)(x+3) = 0$$

$$x = 3 \quad \text{or} \quad x = -3$$

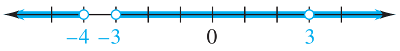


FIGURE 8

The intervals determined by the three (different) solutions are $(-\infty, -4)$, $(-4, -3)$, $(-3, 3)$, and $(3, \infty)$. Testing a number from each interval in the given inequality shows that the solution is

$$(-\infty, -4) \cup (-3, 3) \cup (3, \infty),$$

as shown in Figure 8. For this example, none of the endpoints are part of the solution because $x = 3$ and $x = -3$ make the denominator zero and $x = -4$ produces an equality.

YOUR TURN 3 Solve $\dfrac{k^2 - 35}{k} \ge 2.$

TRY YOUR TURN 3

R.5 EXERCISES

Write each expression in interval notation. Graph each interval.

1. $x < 4$

2. $x \geq -3$

3. $1 \leq x < 2$

4. $-2 \leq x \leq 3$

5. $-9 > x$

6. $6 \leq x$

Using the variable x, write each interval as an inequality.

7. $[-7, -3]$

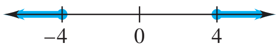
8. $[4, 10)$

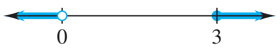
9. $(-\infty, -1]$

10. $(3, \infty)$

11. 

12. 

13. 

14. 

Solve each inequality and graph the solution.

15. $6p + 7 \leq 19$

16. $6k - 4 < 3k - 1$

17. $m - (3m - 2) + 6 < 7m - 19$

18. $-2(3y - 8) \geq 5(4y - 2)$

19. $3p - 1 < 6p + 2(p - 1)$

20. $x + 5(x + 1) > 4(2 - x) + x$

21. $-11 < y - 7 < -1$

22. $8 \leq 3r + 1 \leq 13$

23. $-2 < \dfrac{1 - 3k}{4} \leq 4$

24. $-1 \leq \dfrac{5y + 2}{3} \leq 4$

25. $\dfrac{3}{5}(2p + 3) \geq \dfrac{1}{10}(5p + 1)$

26. $\dfrac{8}{3}(z - 4) \leq \dfrac{2}{9}(3z + 2)$

Solve each inequality. Graph each solution.

27. $(m - 3)(m + 5) < 0$

28. $(t + 6)(t - 1) \geq 0$

29. $y^2 - 3y + 2 < 0$

30. $2k^2 + 7k - 4 > 0$

31. $x^2 - 16 > 0$

32. $2k^2 - 7k - 15 \leq 0$

33. $x^2 - 4x \geq 5$

34. $10r^2 + r \leq 2$

35. $3x^2 + 2x > 1$

36. $3a^2 + a > 10$

37. $9 - x^2 \leq 0$

38. $p^2 - 16p > 0$

39. $x^3 - 4x \geq 0$

40. $x^3 + 7x^2 + 12x \leq 0$

41. $2x^3 - 14x^2 + 12x < 0$

42. $3x^3 - 9x^2 - 12x > 0$

Solve each inequality.

43. $\dfrac{m - 3}{m + 5} \leq 0$

44. $\dfrac{r + 1}{r - 1} > 0$

45. $\dfrac{k - 1}{k + 2} > 1$

46. $\dfrac{a - 5}{a + 2} < -1$

47. $\dfrac{2y + 3}{y - 5} \leq 1$

48. $\dfrac{a + 2}{3 + 2a} \leq 5$

49. $\dfrac{2k}{k - 3} \leq \dfrac{4}{k - 3}$

50. $\dfrac{5}{p + 1} > \dfrac{12}{p + 1}$

51. $\dfrac{2x}{x^2 - x - 6} \geq 0$

52. $\dfrac{8}{p^2 + 2p} > 1$

53. $\dfrac{z^2 + z}{z^2 - 1} \geq 3$

54. $\dfrac{a^2 + 2a}{a^2 - 4} \leq 2$

YOUR TURN ANSWERS

1. $z < -9/2$ **2.** $[-2/3, 6]$ **3.** $[-5, 0) \cup [7, \infty)$

R.6 Exponents

Integer Exponents

Recall that $a^2 = a \cdot a$, while $a^3 = a \cdot a \cdot a$, and so on. In this section, a more general meaning is given to the symbol a^n.

Definition of Exponent

If n is a natural number, then

$$a^n = a \cdot a \cdot a \cdot \cdots \cdot a,$$

where a appears as a factor n times.

In the expression a^n, the power n is the **exponent** and a is the **base**. This definition can be extended by defining a^n for zero and negative integer values of n.

Zero and Negative Exponents

If a is any nonzero real number, and if n is a positive integer, then

$$a^0 = 1 \quad \text{and} \quad a^{-n} = \frac{1}{a^n}.$$

(The symbol 0^0 is meaningless.)

EXAMPLE 1 Exponents

(a) $6^0 = 1$

(b) $(-9)^0 = 1$

(c) $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

(d) $9^{-1} = \frac{1}{9^1} = \frac{1}{9}$

(e) $\left(\frac{3}{4}\right)^{-1} = \frac{1}{(3/4)^1} = \frac{1}{3/4} = \frac{4}{3}$

TRY YOUR TURN 1

YOUR TURN 1 Find $\left(\frac{2}{3}\right)^{-3}$.

The following properties follow from the definitions of exponents given above.

Properties of Exponents

For any integers m and n, and any real numbers a and b for which the following exist:

1. $a^m \cdot a^n = a^{m+n}$
2. $\frac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $(ab)^m = a^m \cdot b^m$
5. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Note that $(-a)^n = a^n$ if n is an even integer, but $(-a)^n = -a^n$ if n is an odd integer.

EXAMPLE 2 Simplifying Exponential Expressions

Use the properties of exponents to simplify each expression. Leave answers with positive exponents. Assume that all variables represent positive real numbers.

(a) $7^4 \cdot 7^6 = 7^{4+6} = 7^{10}$ (or 282,475,249) — Property 1

(b) $\frac{9^{14}}{9^6} = 9^{14-6} = 9^8$ (or 43,046,721) — Property 2

(c) $\frac{r^9}{r^{17}} = r^{9-17} = r^{-8} = \frac{1}{r^8}$ — Property 2

(d) $(2m^3)^4 = 2^4 \cdot (m^3)^4 = 16m^{12}$ — Properties 3 and 4

(e) $(3x)^4 = 3^4 \cdot x^4 = 81x^4$ — Property 4

(f) $\left(\frac{x^2}{y^3}\right)^6 = \frac{(x^2)^6}{(y^3)^6} = \frac{x^{2\cdot 6}}{y^{3\cdot 6}} = \frac{x^{12}}{y^{18}}$ — Properties 3 and 5

(g) $\frac{a^{-3}b^5}{a^4b^{-7}} = \frac{b^{5-(-7)}}{a^{4-(-3)}} = \frac{b^{5+7}}{a^{4+3}} = \frac{b^{12}}{a^7}$ — Property 2

(h) $p^{-1} + q^{-1} = \frac{1}{p} + \frac{1}{q}$ Definition of a^{-n}.

$= \frac{1}{p} \cdot \frac{q}{q} + \frac{1}{q} \cdot \frac{p}{p}$ Get common denominator.

$= \frac{q}{pq} + \frac{p}{pq} = \frac{p + q}{pq}$ Add.

(i) $\frac{x^{-2} - y^{-2}}{x^{-1} - y^{-1}} = \frac{\frac{1}{x^2} - \frac{1}{y^2}}{\frac{1}{x} - \frac{1}{y}}$ Definition of a^{-n}

$= \frac{\frac{y^2 - x^2}{x^2y^2}}{\frac{y - x}{xy}}$ Get common denominators and combine terms.

$= \frac{y^2 - x^2}{x^2y^2} \cdot \frac{xy}{y - x}$ Invert and multiply.

$= \frac{(y - x)(y + x)}{x^2y^2} \cdot \frac{xy}{y - x}$ Factor.

$= \frac{x + y}{xy}$ Simplify. TRY YOUR TURN 2

YOUR TURN 2 Simplify $\left(\frac{y^2z^{-4}}{y^{-3}z^4}\right)^{-2}$.

CAUTION If Example 2(e) were written $3x^4$, the properties of exponents would not apply. When no parentheses are used, the exponent refers only to the factor closest to it. Also notice in Examples 2(c), 2(g), 2(h), and 2(i) that a negative exponent does *not* indicate a negative number.

Roots For *even* values of n and nonnegative values of a, the expression $a^{1/n}$ is defined to be the **positive *n*th root** of a or the **principal *n*th root** of a. For example, $a^{1/2}$ denotes the positive second root, or **square root**, of a, while $a^{1/4}$ is the positive fourth root of a. When n is *odd*, there is only one nth root, which has the same sign as a. For example, $a^{1/3}$, the **cube root** of a, has the same sign as a. By definition, if $b = a^{1/n}$, then $b^n = a$. On a calculator, a number is raised to a power using a key labeled x^y, y^x, or $\wedge$. For example, to take the fourth root of 6 on a TI-84 Plus C calculator, enter $6 \wedge (1/4)$ to get the result 1.56508458.

EXAMPLE 3 Calculations with Exponents

(a) $121^{1/2} = 11$, since 11 is positive and $11^2 = 121$.

(b) $625^{1/4} = 5$, since $5^4 = 625$.

(c) $256^{1/4} = 4$

(d) $64^{1/6} = 2$

(e) $27^{1/3} = 3$

(f) $(-32)^{1/5} = -2$

(g) $128^{1/7} = 2$

(h) $(-49)^{1/2}$ is not a real number.

TRY YOUR TURN 3

YOUR TURN 3 Find $125^{1/3}$.

Rational Exponents In the following definition, the domain of an exponent is extended to include all rational numbers.

Definition of $a^{m/n}$

For all real numbers a for which the indicated roots exist, and for any rational number m/n,

$$a^{m/n} = (a^{1/n})^m.$$

EXAMPLE 4 Calculations with Exponents

YOUR TURN 4 Find $16^{-3/4}$.

(a) $27^{2/3} = (27^{1/3})^2 = 3^2 = 9$

(b) $32^{2/5} = (32^{1/5})^2 = 2^2 = 4$

(c) $64^{4/3} = (64^{1/3})^4 = 4^4 = 256$

(d) $25^{3/2} = (25^{1/2})^3 = 5^3 = 125$

TRY YOUR TURN 4

NOTE $27^{2/3}$ could also be evaluated as $(27^2)^{1/3}$, but this is more difficult to perform without a calculator because it involves squaring 27 and then taking the cube root of this large number. On the other hand, when we evaluate it as $(27^{1/3})^2$, we know that the cube root of 27 is 3 without using a calculator, and squaring 3 is easy.

All the properties for integer exponents given in this section also apply to any rational exponent on a nonnegative real-number base.

EXAMPLE 5 Simplifying Exponential Expressions

YOUR TURN 5 Simplify $\left(\dfrac{x^{1/2}x^4}{x^{3/2}}\right)^{1/3}$.

(a) $\dfrac{y^{1/3}y^{5/3}}{y^3} = \dfrac{y^{1/3+5/3}}{y^3} = \dfrac{y^2}{y^3} = y^{2-3} = y^{-1} = \dfrac{1}{y}$

(b) $m^{2/3}(m^{7/3} + 2m^{1/3}) = m^{2/3+7/3} + 2m^{2/3+1/3} = m^3 + 2m$

(c) $\left(\dfrac{m^7n^{-2}}{m^{-5}n^2}\right)^{1/4} = \left(\dfrac{m^{7-(-5)}}{n^{2-(-2)}}\right)^{1/4} = \left(\dfrac{m^{12}}{n^4}\right)^{1/4} = \dfrac{(m^{12})^{1/4}}{(n^4)^{1/4}} = \dfrac{m^{12/4}}{n^{4/4}} = \dfrac{m^3}{n}$

TRY YOUR TURN 5

In calculus, it is often necessary to factor expressions involving fractional exponents.

EXAMPLE 6 Simplifying Exponential Expressions

Factor out the smallest power of the variable, assuming all variables represent positive real numbers.

(a) $4m^{1/2} + 3m^{3/2}$

SOLUTION The smallest exponent is $1/2$. Factoring out $m^{1/2}$ yields

$$\begin{aligned} 4m^{1/2} + 3m^{3/2} &= m^{1/2}(4m^{1/2-1/2} + 3m^{3/2-1/2}) \\ &= m^{1/2}(4 + 3m). \end{aligned}$$

Check this result by multiplying $m^{1/2}$ by $4 + 3m$.

(b) $9x^{-2} - 6x^{-3}$

SOLUTION The smallest exponent here is -3. Since 3 is a common numerical factor, factor out $3x^{-3}$.

$$9x^{-2} - 6x^{-3} = 3x^{-3}(3x^{-2-(-3)} - 2x^{-3-(-3)}) = 3x^{-3}(3x - 2)$$

Check by multiplying. The factored form can be written without negative exponents as

$$\frac{3(3x-2)}{x^3}.$$

(c) $(x^2 + 5)(3x - 1)^{-1/2}(2) + (3x - 1)^{1/2}(2x)$

SOLUTION There is a common factor of 2. Also, $(3x - 1)^{-1/2}$ and $(3x - 1)^{1/2}$ have a common factor. Always factor out the quantity to the *smallest* exponent. Here $-1/2 < 1/2$, so the common factor is $2(3x - 1)^{-1/2}$ and the factored form is

$$2(3x - 1)^{-1/2}[(x^2 + 5) + (3x - 1)x] = 2(3x - 1)^{-1/2}(4x^2 - x + 5).$$

TRY YOUR TURN 6

YOUR TURN 6 Factor $5z^{1/3} + 4z^{-2/3}$.

R.6 EXERCISES

Evaluate each expression. Write all answers without exponents.

1. 8^{-2}

2. 3^{-4}

3. 5^0

4. $\left(-\frac{3}{4}\right)^0$

5. $-(-3)^{-2}$

6. $-(-3^{-2})$

7. $\left(\frac{1}{6}\right)^{-2}$

8. $\left(\frac{4}{3}\right)^{-3}$

Simplify each expression. Assume that all variables represent positive real numbers. Write answers with only positive exponents.

9. $\frac{4^{-2}}{4}$

10. $\frac{8^9 \cdot 8^{-7}}{8^{-3}}$

11. $\frac{10^8 \cdot 10^{-10}}{10^4 \cdot 10^2}$

12. $\left(\frac{7^{-12} \cdot 7^3}{7^{-8}}\right)^{-1}$

13. $\frac{x^4 \cdot x^3}{x^5}$

14. $\frac{y^{10} \cdot y^{-4}}{y^6}$

15. $\frac{(4k^{-1})^2}{2k^{-5}}$

16. $\frac{(3z^2)^{-1}}{z^5}$

17. $\frac{3^{-1} \cdot x \cdot y^2}{x^{-4} \cdot y^5}$

18. $\frac{5^{-2}m^2y^{-2}}{5^2m^{-1}y^{-2}}$

19. $\left(\frac{a^{-1}}{b^2}\right)^{-3}$

20. $\left(\frac{c^3}{7d^{-2}}\right)^{-2}$

Simplify each expression, writing the answer as a single term without negative exponents.

21. $a^{-1} + b^{-1}$

22. $b^{-2} - a$

23. $\frac{2n^{-1} - 2m^{-1}}{m + n^2}$

24. $\left(\frac{m}{3}\right)^{-1} + \left(\frac{n}{2}\right)^{-2}$

25. $(x^{-1} - y^{-1})^{-1}$

26. $(x \cdot y^{-1} - y^{-2})^{-2}$

Write each number without exponents.

27. $121^{1/2}$

28. $27^{1/3}$

29. $32^{2/5}$

30. $-125^{2/3}$

31. $\left(\frac{36}{144}\right)^{1/2}$

32. $\left(\frac{64}{27}\right)^{1/3}$

33. $8^{-4/3}$

34. $625^{-1/4}$

35. $\left(\frac{27}{64}\right)^{-1/3}$

36. $\left(\frac{121}{100}\right)^{-3/2}$

Simplify each expression. Write all answers with only positive exponents. Assume that all variables represent positive real numbers.

37. $3^{2/3} \cdot 3^{4/3}$

38. $27^{2/3} \cdot 27^{-1/3}$

39. $\frac{4^{9/4} \cdot 4^{-7/4}}{4^{-10/4}}$

40. $\frac{3^{-5/2} \cdot 3^{3/2}}{3^{7/2} \cdot 3^{-9/2}}$

41. $\left(\frac{x^6y^{-3}}{x^{-2}y^5}\right)^{1/2}$

42. $\left(\frac{a^{-7}b^{-1}}{b^{-4}a^2}\right)^{1/3}$

43. $\frac{7^{-1/3} \cdot 7r^{-3}}{7^{2/3} \cdot (r^{-2})^2}$

44. $\frac{12^{3/4} \cdot 12^{5/4} \cdot y^{-2}}{12^{-1} \cdot (y^{-3})^{-2}}$

45. $\frac{3k^2 \cdot (4k^{-3})^{-1}}{4^{1/2} \cdot k^{7/2}}$

46. $\frac{8p^{-3} \cdot (4p^2)^{-2}}{p^{-5}}$

47. $\frac{a^{4/3} \cdot b^{1/2}}{a^{2/3} \cdot b^{-3/2}}$

48. $\frac{x^{3/2} \cdot y^{4/5} \cdot z^{-3/4}}{x^{5/3} \cdot y^{-6/5} \cdot z^{1/2}}$

49. $\frac{k^{-3/5} \cdot h^{-1/3} \cdot t^{2/5}}{k^{-1/5} \cdot h^{-2/3} \cdot t^{1/5}}$

50. $\frac{m^{7/3} \cdot n^{-2/5} \cdot p^{3/8}}{m^{-2/3} \cdot n^{3/5} \cdot p^{-5/8}}$

Factor each expression.

51. $3x^3(x^2 + 3x)^2 - 15x(x^2 + 3x)^2$

52. $6x(x^3 + 7)^2 - 6x^2(3x^2 + 5)(x^3 + 7)$

53. $10x^3(x^2 - 1)^{-1/2} - 5x(x^2 - 1)^{1/2}$

54. $9(6x + 2)^{1/2} + 3(9x - 1)(6x + 2)^{-1/2}$

55. $x(2x + 5)^2(x^2 - 4)^{-1/2} + 2(x^2 - 4)^{1/2}(2x + 5)$

56. $(4x^2 + 1)^2(2x - 1)^{-1/2} + 16x(4x^2 + 1)(2x - 1)^{1/2}$

YOUR TURN ANSWERS

1. $27/8$ **2.** z^{16}/y^{10} **3.** 5

4. $1/8$ **5.** x **6.** $z^{-2/3}(5z + 4)$

R.7 Radicals

We have defined $a^{1/n}$ as the positive or principal nth root of a for appropriate values of a and n. An alternative notation for $a^{1/n}$ uses radicals.

Radicals

If n is an even natural number and $a > 0$, or n is an odd natural number, then

$$a^{1/n} = \sqrt[n]{a}.$$

The symbol $\sqrt[n]{\ }$ is a **radical sign**, the number a is the **radicand**, and n is the **index** of the radical. The familiar symbol $\sqrt{a}$ is used instead of $\sqrt[2]{a}$.

EXAMPLE 1 Radical Calculations

(a) $\sqrt[4]{16} = 16^{1/4} = 2$

(b) $\sqrt[5]{-32} = -2$

(c) $\sqrt[3]{1000} = 10$

(d) $\sqrt[6]{\dfrac{64}{729}} = \dfrac{2}{3}$

With $a^{1/n}$ written as $\sqrt[n]{a}$, the expression $a^{m/n}$ also can be written using radicals.

$$a^{m/n} = (\sqrt[n]{a})^m \quad \text{or} \quad a^{m/n} = \sqrt[n]{a^m}$$

The following properties of radicals depend on the definitions and properties of exponents.

Properties of Radicals

For all real numbers a and b and natural numbers m and n such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers:

1. $(\sqrt[n]{a})^n = a$
2. $\sqrt[n]{a^n} = \begin{cases} |a| & \text{if } n \text{ is even} \\ a & \text{if } n \text{ is odd} \end{cases}$
3. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$
4. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}} \quad (b \neq 0)$
5. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

Property 3 can be used to simplify certain radicals. For example, since $48 = 16 \cdot 3$,

$$\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{16} \cdot \sqrt{3} = 4\sqrt{3}.$$

To some extent, simplification is in the eye of the beholder, and $\sqrt{48}$ might be considered as simple as $4\sqrt{3}$. In this textbook, we will consider an expression to be simpler when we have removed as many factors as possible from under the radical.

EXAMPLE 2 Radical Calculations

(a) $\sqrt{1000} = \sqrt{100 \cdot 10} = \sqrt{100} \cdot \sqrt{10} = 10\sqrt{10}$

(b) $\sqrt{128} = \sqrt{64 \cdot 2} = 8\sqrt{2}$

(c) $\sqrt{2} \cdot \sqrt{18} = \sqrt{2 \cdot 18} = \sqrt{36} = 6$

(d) $\sqrt[3]{54} = \sqrt[3]{27 \cdot 2} = \sqrt[3]{27} \cdot \sqrt[3]{2} = 3\sqrt[3]{2}$

(e) $\sqrt{288m^5} = \sqrt{144 \cdot m^4 \cdot 2m} = 12m^2\sqrt{2m}$

(f)
$$\begin{aligned} 2\sqrt{18} - 5\sqrt{32} &= 2\sqrt{9 \cdot 2} - 5\sqrt{16 \cdot 2} \\ &= 2\sqrt{9} \cdot \sqrt{2} - 5\sqrt{16} \cdot \sqrt{2} \\ &= 2(3)\sqrt{2} - 5(4)\sqrt{2} = -14\sqrt{2} \end{aligned}$$

YOUR TURN 1 Simplify $\sqrt{28x^9y^5}$.

(g) $\sqrt{x^5} \cdot \sqrt[3]{x^5} = x^{5/2} \cdot x^{5/3} = x^{5/2+5/3} = x^{25/6} = \sqrt[6]{x^{25}} = x^4\sqrt[6]{x}$

TRY YOUR TURN 1

When simplifying a square root, keep in mind that $\sqrt{x}$ is nonnegative by definition. Also, $\sqrt{x^2}$ is not x, but $|x|$, the **absolute value of *x***, defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

For example, $\sqrt{(-5)^2} = |-5| = 5$. It is correct, however, to simplify $\sqrt{x^4} = x^2$. We need not write $|x^2|$ because x^2 is always nonnegative.

EXAMPLE 3 Simplifying by Factoring

Simplify $\sqrt{m^2 - 4m + 4}$.

SOLUTION Factor the polynomial as $m^2 - 4m + 4 = (m-2)^2$. Then by property 2 of radicals and the definition of absolute value,

$$\sqrt{(m-2)^2} = |m-2| = \begin{cases} m - 2 & \text{if } m - 2 \geq 0 \\ -(m-2) = 2 - m & \text{if } m - 2 < 0. \end{cases}$$

CAUTION Avoid the common error of writing $\sqrt{a^2 + b^2}$ as $\sqrt{a^2} + \sqrt{b^2}$. We must add a^2 and b^2 *before* taking the square root. For example, $\sqrt{16 + 9} = \sqrt{25} = 5$, *not* $\sqrt{16} + \sqrt{9} = 4 + 3 = 7$. This idea applies as well to higher roots. For example, in general,

$$\sqrt[3]{a^3 + b^3} \neq \sqrt[3]{a^3} + \sqrt[3]{b^3},$$

$$\sqrt[4]{a^4 + b^4} \neq \sqrt[4]{a^4} + \sqrt[4]{b^4}.$$

Also, $$\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}.$$

Rationalizing Denominators

The next example shows how to *rationalize* (remove all radicals from) the denominator in an expression containing radicals.

EXAMPLE 4 Rationalizing Denominators

Simplify each expression by rationalizing the denominator.

(a) $\dfrac{4}{\sqrt{3}}$

SOLUTION To rationalize the denominator, multiply by $\sqrt{3}/\sqrt{3}$ (or 1) so the denominator of the product is a rational number.

$$\frac{4}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{4\sqrt{3}}{3} \qquad \sqrt{3} \cdot \sqrt{3} = \sqrt{9} = 3$$

(b) $\dfrac{2}{\sqrt[3]{x}}$

SOLUTION Here, we need a perfect cube under the radical sign to rationalize the denominator. Multiplying by $\sqrt[3]{x^2}/\sqrt[3]{x^2}$ gives

$$\frac{2}{\sqrt[3]{x}} \cdot \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}} = \frac{2\sqrt[3]{x^2}}{\sqrt[3]{x^3}} = \frac{2\sqrt[3]{x^2}}{x}.$$

(c) $\dfrac{1}{1 - \sqrt{2}}$

SOLUTION The best approach here is to multiply both numerator and denominator by the number $1 + \sqrt{2}$. The expressions $1 + \sqrt{2}$ and $1 - \sqrt{2}$ are conjugates,* and their product is $1^2 - (\sqrt{2})^2 = 1 - 2 = -1$. Thus,

$$\frac{1}{1 - \sqrt{2}} = \frac{1(1 + \sqrt{2})}{(1 - \sqrt{2})(1 + \sqrt{2})} = \frac{1 + \sqrt{2}}{1 - 2} = -1 - \sqrt{2}.$$

TRY YOUR TURN 2

YOUR TURN 2 Rationalize the denominator in $\dfrac{5}{\sqrt{x} - \sqrt{y}}$.

Sometimes it is advantageous to rationalize the *numerator* of a rational expression. The following example arises in calculus when evaluating a *limit.*

EXAMPLE 5 Rationalizing Numerators

Rationalize each numerator.

(a) $\dfrac{\sqrt{x} - 3}{x - 9}$.

SOLUTION Multiply the numerator and denominator by the conjugate of the numerator, $\sqrt{x} + 3$.

$$\begin{aligned} \frac{\sqrt{x} - 3}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3} &= \frac{(\sqrt{x})^2 - 3^2}{(x - 9)(\sqrt{x} + 3)} \qquad (a - b)(a + b) = a^2 - b^2 \\ &= \frac{x - 9}{(x - 9)(\sqrt{x} + 3)} \\ &= \frac{1}{\sqrt{x} + 3} \end{aligned}$$

*If a and b are real numbers, the *conjugate* of $a + b$ is $a - b$.

(b) $\dfrac{\sqrt{3}+\sqrt{x+3}}{\sqrt{3}-\sqrt{x+3}}$

SOLUTION Multiply the numerator and denominator by the conjugate of the numerator, $\sqrt{3}-\sqrt{x+3}$.

$$\frac{\sqrt{3}+\sqrt{x+3}}{\sqrt{3}-\sqrt{x+3}}\cdot\frac{\sqrt{3}-\sqrt{x+3}}{\sqrt{3}-\sqrt{x+3}}=\frac{3-(x+3)}{3-2\sqrt{3}\sqrt{x+3}+(x+3)}$$
$$=\frac{-x}{6+x-2\sqrt{3(x+3)}}$$

YOUR TURN 3 Rationalize the numerator in $\dfrac{4+\sqrt{x}}{16-x}$.

TRY YOUR TURN 3

R.7 EXERCISES

Simplify each expression by removing as many factors as possible from under the radical. Assume that all variables represent positive real numbers.

1. $\sqrt[3]{125}$
2. $\sqrt[4]{1296}$
3. $\sqrt[5]{-3125}$
4. $\sqrt{50}$
5. $\sqrt{2000}$
6. $\sqrt{32y^5}$
7. $\sqrt{27}\cdot\sqrt{3}$
8. $\sqrt{2}\cdot\sqrt{32}$
9. $7\sqrt{2}-8\sqrt{18}+4\sqrt{72}$
10. $4\sqrt{3}-5\sqrt{12}+3\sqrt{75}$
11. $4\sqrt{7}-\sqrt{28}+\sqrt{343}$
12. $3\sqrt{28}-4\sqrt{63}+\sqrt{112}$
13. $\sqrt[3]{2}-\sqrt[3]{16}+2\sqrt[3]{54}$
14. $2\sqrt[3]{5}-4\sqrt[3]{40}+3\sqrt[3]{135}$
15. $\sqrt{2x^3y^2z^4}$
16. $\sqrt{160r^7s^9t^{12}}$
17. $\sqrt[3]{128x^3y^8z^9}$
18. $\sqrt[4]{x^8y^7z^{11}}$
19. $\sqrt{a^3b^5}-2\sqrt{a^7b^3}+\sqrt{a^3b^9}$
20. $\sqrt{p^7q^3}-\sqrt{p^5q^9}+\sqrt{p^9q}$
21. $\sqrt{a}\cdot\sqrt[3]{a}$
22. $\sqrt{b^3}\cdot\sqrt[4]{b^3}$

Simplify each root, if possible.

23. $\sqrt{16-8x+x^2}$
24. $\sqrt{9y^2+30y+25}$
25. $\sqrt{4-25z^2}$
26. $\sqrt{9k^2+h^2}$

Rationalize each denominator. Assume that all radicands represent positive real numbers.

27. $\dfrac{5}{\sqrt{7}}$
28. $\dfrac{5}{\sqrt{10}}$
29. $\dfrac{-3}{\sqrt{12}}$
30. $\dfrac{4}{\sqrt{8}}$
31. $\dfrac{3}{1-\sqrt{2}}$
32. $\dfrac{5}{2-\sqrt{6}}$
33. $\dfrac{6}{2+\sqrt{2}}$
34. $\dfrac{\sqrt{5}}{\sqrt{5}+\sqrt{2}}$
35. $\dfrac{1}{\sqrt{r}-\sqrt{3}}$
36. $\dfrac{5}{\sqrt{m}-\sqrt{5}}$
37. $\dfrac{y-5}{\sqrt{y}-\sqrt{5}}$
38. $\dfrac{\sqrt{z}-1}{\sqrt{z}-\sqrt{5}}$
39. $\dfrac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}}$
40. $\dfrac{\sqrt{p}+\sqrt{p^2-1}}{\sqrt{p}-\sqrt{p^2-1}}$

Rationalize each numerator. Assume that all radicands represent positive real numbers.

41. $\dfrac{1+\sqrt{2}}{2}$
42. $\dfrac{3-\sqrt{3}}{6}$
43. $\dfrac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}}$
44. $\dfrac{\sqrt{p}-\sqrt{p-2}}{\sqrt{p}}$

YOUR TURN ANSWERS

1. $2x^4y^2\sqrt{7xy}$ **2.** $5(\sqrt{x}+\sqrt{y})/(x-y)$ **3.** $1/(4-\sqrt{x})$

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1 Linear Functions

Over short time intervals, many changes in the economy are well modeled by linear functions. In an exercise in the first section of this chapter, we will examine a linear model that predicts the number of cellular telephone users in the United States. Such predictions are important tools for cellular telephone company executives and planners.